FEASIBLE OBJECTIVE CONFIDENCE BOUNDS FOR SYSTEM RELIABILITY

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M. VERNON JOHNS, JR.

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ABSTRACT

The basic problem of determining objective (frequentistic) confidence bounds for the reliability of a series system based on failure data from tests of the independent components is addressed. The notion of confidence bounds based on orderings imposed on the sample space is exploited, and certain optimality considerations are incorporated. Advantage is taken of the simplifications resulting from the use of the Poisson approximation for data from highly reliable components. Tables of exact confidence bounds are produced for the case of two-component systems. These bounds are computed using sample orderings generated sequentially by a two-stage, prospective optimization procedure. A generalization of the Lindstrom-Madden technique is proposed for using the tables to find confidence bounds for systems consisting of more than two components with differing sample sizes.

Key Words: Reliability, series-system, confidence bounds, Poisson approximation, Lindstrom-Madden, sample orderings.

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1. Introduction

Certainly one of the most basic statistical problems in the assessment of system reliability is that of determining a confidence bound on the reliability of a series system based on component failure data. The continuing appearance of papers concerned with this subject (e.g., Harris and Soms 1981, Butcher et al. 1978, and Winterbottom 1980) testifies to the elusiveness of solutions which are both practically feasible and acceptably precise. The present paper deals with the case of systems characterized by high intrinsic reliability (>90%) where the use of the Poisson approximation for the binomial distributions of component failure data does not introduce appreciable error. The emphasis is on objective, frequentistic confidence bounds which avoid the uncertainties of interpretation associated with posterior bounds obtained by Bayesian methods. The use of the Poisson distribution provides several advantages. It easily leads to valid results for cases involving zero observed failures for some components where maximum likelihood methods and other proposed approximations tend to break down (see, e.g., Madansky 1965 and Mann et al. 1974). It also permits the pooling of failure data for different components having the same test sample sizes. This potential reduction in the effective number of system components enhances the utility of tabulated bounds such as those presented here.

Because of the structure of the problem, universally optimal confidence bounds for system reliability (i.e., "uniformly most accurate" bounds in the sense of Lehmann 1959) do not generally exist. On the other hand, the ideas of Buehler (1957) may be exploited to produce a variety of valid confidence bounds based on total orderings of the sample points. Such bounds are exact in the sense that the desired coverage probability is guaranteed. The construction of good confidence bounds is thus reduced to the selection of suitable orderings imposed on the sample space. This is the approach adopted in the present study. Previous applications of these methods to reliability may be found, for example, in Harris and Soms (1980), Johnson (1969), and Lipow and Riley (1959).

Once the sample ordering approach has been chosen, there remains the problem of determining orderings which lead to confidence bounds which are "good" according to some criterion measuring the size (length) of the confidence region. An early proposal of the present author (Johns 1975) was to generate the ordering by means of a simple function of the observations which was asymptotically equivalent to the maximum likelihood confidence bound. This method guarantees asymptotic optimality when the numbers of component failures observed under testing is large. It was found, however, that the resulting bounds could be noticeably improved for small to moderate numbers of observed failures by more sophisticated methods. Another procedure investigated (Johns 1977) was a sequential method for generating the sample ordering starting at the origin (zero component failures observed) and selecting at each stage as the next sample

point the "adjacent" point producing the largest value for the lower bound on reliability. This method, while intuitively appealing, does not generally produce a "best" ordering as has been suggested by some investigators. In particular, it is improved upon by the method adopted in the present paper.

The non-existence of a unique optimal ordering leaves open the possibility of obtaining a confidence bound which is at least admissible by choosing the sample ordering to minimize the expected length of the confidence interval under some suitable prior distribution. Such a semi-Bayesian approach does not in any way impair the frequentistic interpretation of the confidence bounds obtained from the minimizing ordering. The implementation of such a minimization, while theoretically perfectly possible, turns out to be totally unfeasible computationally except for the earliest part of the ordering generated. Nevertheless, for a class of priors chosen to emulate certain properties of maximum likelihood, fragmentary orderings computed by this method provide a considerable justification for the two-stage "look-ahead" sequential method actually used to generate the tables which are a principal concern of this paper.

Suppose that the series system under consideration consists of k independent components and that the respective probabilities of component failure are $p_i = 1 - q_i$, $i = 1,2,\ldots,k$. The system reliability R is given by

$$R = \prod_{i=1}^{k} q_i = \prod_{i=1}^{k} (1 - p_i) \quad . \tag{1.1}$$

If the observed numbers of failures for the k components are $X_1, X_2, \ldots, X_k$ based on independent tests with corresponding sample sizes $n_1, n_2, \ldots, n_k$, we let $\lambda_i = n_i p_i$, $i = 1,2,\ldots,k$, so that

$$R = \prod_{i=1}^{k} (1 - \lambda_i/n_i) \cong 1 - \sum_{i=1}^{k} \lambda_i/n_i \quad . \tag{1.2}$$

The approximation on the right will be best when the p_i's $(= \lambda_i/n_i)$ are all small which is just the case where R is close to one and the Poisson approximation for the distributions of the X_i's is valid.

It will be convenient to express the problem in a canonical form by introducing some further notation. Let $c = \Sigma_{i=1}^{k} 1/n_i$ and $a_i = 1/cn_i$, $i = 1,2,\ldots,k$. Then letting $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ and

$$\theta(\lambda) = \sum_{i=1}^{k} a_i \lambda_i \quad , \tag{1.3}$$

we have

$$R \cong 1 - c \sum_{i=1}^{k} a_i \lambda_i = 1 - c\theta(\lambda) \quad . \tag{1.4}$$

We shall assume henceforth that the components are indexed so that $n_1 \geq n_2 \geq \ldots \geq n_k$ which implies that $a_1 \leq a_2 \leq \ldots \leq a_k$. The problem of finding a lower confidence bound for R is thus reduced to that of finding an <u>upper</u> confidence bound for $\theta(\lambda)$. The fact that $\theta(\lambda)$ is a convex combination of the λ_i's facilitates tabulation of the bounds by reducing the number of classification variables by one. In principle, confidence bounds for R could be constructed directly without introducing the approximation (1.2). Such an approach would,

however, eliminate the possibility of constructing useful tables of bounds, since separate entries would be required for every configuration of $(n_1, n_2, \ldots, n_k)$.

The vector of observations is $X = (X_1, X_2, \ldots, X_k)$ so that the sample space on which a total ordering must be imposed consists of all vectors $x = (x_1, x_2, \ldots, x_k)$ where the x_i's are non-negative integers. It seems reasonable to confine attention to orderings which are consistent with the natural partial ordering induced by dominence (see Section 2), and we shall do so. In Section 2 it is shown that the best upper confidence bound for $\theta(\lambda)$ with confidence coefficient $1 - \alpha$ which is monotone in a prescribed total ordering (designated by the relation $\preccurlyeq$) is given by

$$t(x) = \sup_{\lambda \varepsilon S^*_\alpha(x)} \theta(\lambda) \quad , \tag{1.5}$$

where $S^*_\alpha(x) = \{\lambda : P_\lambda\{X \preccurlyeq x\} = \alpha\}$. From (1.4) the lower confidence bound for system reliability is then given by

$$r(x) = 1 - ct(x) \quad . \tag{1.6}$$

Because of the additive property of independent Poisson observations the effective number of components in the system may be reduced if some sample sizes are equal (or nearly equal) as follows: Suppose $n_1 = n_2 = \ldots = n_r = n$ for some r, $2 \leq r < k$. Then $a_1 = a_2 = \ldots = a_r = a^*$ (say) and letting $\lambda^* = \Sigma_{i=1}^{r} \lambda_i$ we have

$$\theta(\lambda) = a^*\lambda^* + \sum_{i=r+1}^{k} a_i\lambda_i \quad . \tag{1.7}$$

Now $X^* = \Sigma_{i=1}^{r} X_i$ has a Poisson distribution with parameter λ^* so that the right hand side of (1.7) has the correct form for the case of dimension $k^* = k - r + 1$ except that the coefficients must each be divided by $c_0 = a^* + a_{r+1} + \ldots + a_k$ to preserve convexity. The confidence bound for the k^* dimentional case based on $X^*, X_{r+1}, \ldots, X_k$ and the normalized coefficients may then be multipled by c_0 to obtain the bound for the original $\theta(\lambda)$ given by (1.7). Further reductions may be made in the same way if several groups of components have common sample sizes. If $n_1, n_2, \ldots, n_r$ are only approximately equal, their average may be used for n in the above calculations to obtain an approximate bound.

If all components are subjected to the same number of trials, we may take k equal to one and the problem reduces to the familiar one of finding an upper confidence bound for a single Poisson parameter.

The component failure data may be developed through independent testing of the components, or through testing of the complete system with the assignment of failures to the appropriate components.

Even in the latter case component sample sizes may differ if components are redesigned during the course of testing so that the trials and failures observed prior to redesign are not relevant to the reliability of the final version of the system.

It is worth noting that the same formal confidence bound problem arises from a somewhat different and more restrictive model involving continuous time sampling and exponentially distributed failure times. Thus, if we assume that the time to failure of the i-th component has the exponential probability density function $\mu_i e^{-\mu_i t}$, $\mu_i > 0$, $t > 0$, and if the fixed aggregate testing time for the i-th component is given by τ_i and the total number of failures observed is X_i, $i = 1,2,\ldots,k$, then the X_i's which are sufficient statistics have independent Poisson distributions with corresponding parameters $\lambda_i = \mu_i \tau_i$. Such test data would be generated, for example, if a single unit of each component were placed on test and repaired whenever a failure occurred until the total test time for each component reached its corresponding limit, τ_i. The system reliability for a single duty cycle (consisting of operation for a unit interval of time) is given by

$$R = \prod_{i=1}^{k} e^{-\mu_i} = \exp\left[-\prod_{i=1}^{k} \mu_i\right] = \exp\left\{-\sum_{i=1}^{k} \lambda_i/\tau_i\right\} , \qquad (1.8)$$

and the confidence bound problem is essentially the same as the one previously introduced except that no approximations are required.

The general theory of confidence bounds based on sample orderings is discussed in Section 2. In Section 3 the case of systems having two components is considered in detail, and the rational for, and use of, the tables for this case are explained. Section 4 includes suggestions for constructing approximate confidence bounds for cases of systems with $k \geq 3$ by finding approximately equivalent cases with $k = 2$. Use of the maximum likelihood ratio bounds for cases where the data are beyond the limits of the available tables is also discussed.

2. Bounds and Orderings: Generalities

The idea of using sample orderings to generate confidence bounds was first introduced by Buehler (1957) who discussed the validity of the proposed method in the context of a specific reliability problem. Bol'shev and Loginov (1969) discuss the construction of confidence bounds monotone in the sample orderings generated by certain functions of the observations. In the following, which is a revision and extension of Johns (1975), we develop the theory with emphasis on the sample orderings themselves rather than possible generators of the orderings.

To develop the general ideas relating exact confidence bounds to sample orderings it is convenient to introduce a fairly abstract statistical model. Let the sample space $\mathcal{X}$ be endowed with a measurable total ordering relation "$\preccurlyeq$" and let X represent the random outcome of the experiment where the space of possible outcomes is $\mathcal{X}$. Suppose that the possible distributions of X are determined by the family of probability measures P_λ, indexed by λ, an element of the parameter space Λ. Our objective is to find a $1 - \alpha$ level upper confidence bound for a specified real-valued function $\theta(\lambda)$ defined on Λ where the range T of $\theta(\lambda)$ is assumed to be closed and bounded below. The quantity $\alpha \varepsilon (0,1)$ is regarded as fixed throughout. We make the following definitions and assumptions:

Definition D1. For each $x \varepsilon \mathcal{X}$ let

$$S_\alpha(x) = \{\lambda : P_\lambda\{X \preccurlyeq x\} > \alpha\} \quad . \tag{2.1}$$

Definition D2. For each $x \varepsilon \mathcal{X}$ let

$$t(x) = \begin{cases} \sup\{\theta(\lambda) : \lambda \varepsilon S_\alpha(x)\} \ , & \text{if} \quad S_\alpha(x) \quad \text{is non-empty} \ , \\ \inf T, \quad \text{otherwise} \quad . \end{cases} \tag{2.2}$$

Remark 1. By D2 if $\theta(\lambda) > t(x)$, then necessarily $P_\lambda\{X \preccurlyeq x\} \le \alpha$.

Assumption A1. For every subset C of $\mathcal{X}$ having the property that if $x \varepsilon C$ and $y \preccurlyeq x$ then $y \varepsilon C$, there exists an ordered sequence $x_1 \preccurlyeq x_2 \preccurlyeq \dots$ of elements of C such that $C = \bigcup_{n=1}^{\infty}\{x : x \preccurlyeq x_n\}$.

Assumption A2. For each $x \varepsilon \mathcal{X}$, if $\theta(\lambda) = t(x)$, then $P_\lambda\{X \preccurlyeq x\} \le \alpha$.

Remark 2. By D1, D2, and A2 if $\theta(\lambda) = t(x)$, then $\lambda \notin S_\alpha(x)$, i.e., the supremum in D2 is never attained.

We now establish the following propositions.

Proposition P1. The function $t(x)$ is monotone in the ordering on $\mathfrak{X}$.

Proof: If $x, y \in \mathfrak{X}$ and $x \preccurlyeq y$, then $S_\alpha(x) \subset S_\alpha(y)$ (D1) which in turn implies $t(x) \le t(y)$ (D2). $\square$

Proposition P2. Under assumptions A1 and A2 the function $t(x)$ is an upper confidence bound for $\theta(\lambda)$ at level $1 - \alpha$. In particular,

$$P_\lambda\{\theta(\lambda) < t(X)\} \ge 1 - \alpha \quad \text{for all} \quad \lambda \in \Lambda \ . \tag{2.3}$$

Proof: For arbitrary $\lambda_0 \in \Lambda$, let $\theta_0 = \theta(\lambda_0)$ and $C_0 = \{x : t(x) \le \theta_0\}$. The result follows immediately for all λ_0 for which C_0 is empty. Assume that C_0 is non-empty. Then by P1 the set C_0 possesses the property required in A1 for the existence of a sequence $\{x_n\} \subset C_0$ such that $x_n \preccurlyeq x_{n+1}$ for all n, and $C_0 = \bigcup_{n=1}^{\infty} \{x : x \preccurlyeq x_n\}$. This implies that, as $n \to \infty$,

$$P_{\lambda_0}\{X \preccurlyeq x_n\} \uparrow P_{\lambda_0}\{X \in C_0\} \ . \tag{2.4}$$

But by Remark 1 and A2, for all n, $P_{\lambda_0}\{X \preccurlyeq x_n\} < \alpha$. Hence $P_{\lambda_0}\{X \in C_0\} \le \alpha$ and the desired result follows. $\square$

Proposition P3. Under assumptions A1 and A2, if $\tilde{t}(x)$ is any T-valued confidence bound such that $P_\lambda\{\theta(\lambda) < \tilde{t}(X)\} \geq 1 - \alpha$ for all $\lambda \varepsilon \Lambda$, then

(i) $\sup_{y \preccurlyeq x} \tilde{t}(y) \geq t(x)$ for all $x \varepsilon \mathfrak{X}$, and

(ii) if $t(x)$ is monotone in the ordering on $\mathfrak{X}$, then

$\tilde{t}(x) \geq t(x)$ for all $x \varepsilon \mathfrak{X}$.

Proof: First we assume that $\tilde{t}(x)$ is monotone and establish (ii). Suppose there exists an $x' \varepsilon \mathfrak{X}$ such that $\tilde{t}(x') < t(x')$. Then $S_\alpha(x')$ must be non-empty and by Remark 2 following A2, the sup defining $t(x')$ is not attained. Hence there exists a $\lambda' \varepsilon S_\alpha(x')$ such that $\tilde{t}(x') < \theta(\lambda') < t(x')$, and $P_{\lambda'}\{X \preccurlyeq x'\} > \alpha$. Thus by the monotonicty of $\tilde{t}(x)$,

$$P_{\lambda'}\{\tilde{t}(X) \leq \theta(\lambda')\} \geq P_{\lambda'}\{\tilde{t}(X) \leq \tilde{t}(x')\} = P_{\lambda'}\{X \preccurlyeq x'\} > \alpha \quad . \qquad (2.5)$$

This contradicts the hypothesis that $P_\lambda\{\theta(\lambda) < \tilde{t}(X)\} \geq 1 - \alpha$ for all λ and establishes (ii). To show (i) we let $t^*(x) = \sup_{y \preccurlyeq x} \tilde{t}(y)$ and observe that $t^*(x)$ is monotone in the ordering on $\mathfrak{X}$ and $t^*(x) \geq \tilde{t}(x)$ for all $x \varepsilon \mathfrak{X}$. Hence if $\tilde{t}(x)$ is a $1 - \alpha$ confidence bound for $\theta(\lambda)$, so is $t^*(x)$ and applying (ii) to $t^*(x)$ yields (i). $\square$.

In order to specialize these results in the direction of applications we henceforth assume that the parameter λ and the

observation X are both of dimension k, i.e., $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_k)$ and $X = (X_1,X_2,\ldots,X_k)$ where the λ_i's are real and the X_i's are random variables. Without essential loss of generality we assume that Λ contains the positive orthant. Within this framework we make the following additional assumptions:

<u>Assumption A3</u>. The function $\theta(\lambda)$ is continuous and strictly increasing in each of the λ_i's.

<u>Assumption A4</u>. For any $x \in \mathfrak{X}$, $P_\lambda\{X \preccurlyeq x\}$ is continuous in each of the λ_i's.

<u>Proposition P4</u>. Assumptions A3 and A4 imply that Assumption A2 is satisfied.

<u>Proof</u>: Suppose that for some $x' \in \mathfrak{X}$ there exists a $\lambda' \in \Lambda$ such that $\theta(\lambda') = t(x')$ and $P_{\lambda'}\{X \preccurlyeq x'\} > \alpha$. Then by A3 and A4 we can find a $\lambda'' \in \Lambda$ with $\lambda_i'' \geq \lambda_i'$ for all i and $\lambda_{i_o}'' > \lambda_{i_o}''$ for some i_o such that $\theta(\lambda'') > t(x')$ and $P_{\lambda''}\{X \preccurlyeq x'\} > \alpha$. This contradicts D2 and the result follows. $\square$

<u>Corollary C1</u>. If (i) the observations $X_1,X_2,\ldots,X_k$ are independent Poisson random variables with parameters $\lambda_1,\lambda_2,\ldots,\lambda_k$ respectively, and (ii) $\theta(\lambda) = a_1\lambda_1 + a_2\lambda_2 + \ldots + a_k\lambda_k$ where the a_i's are positive, then $t(x)$ given by D2 is an upper confidence bound for $\theta(\lambda)$ at level $1-\alpha$.

Proof: A1 is satisfied for any ordering since $\mathfrak{X}$ is discrete. A3 is clearly satisfied for $\theta(\lambda)$ of the form given by (ii), and A4 is satisfied for Poisson random variables. The desired result follows from P4 and P2. $\square$

The actual computation of the bound $t(x)$ given by D2 is greatly facilitated if $P_\lambda\{X \preccurlyeq x\}$ is monotone in the components of λ. The following proposition gives conditions guaranteeing this property:

Assumption A5. The components of X are independent and for $i = 1,2,\ldots,k$ the distribution of each X_i depends only on the corresponding λ_i and is stochastically increasing in λ_i.

Definition D3. We denote by $(\overset{*}{\preccurlyeq})$ the natural partial ordering of $\mathfrak{X}$ generated by componentwise dominance. That is, $x \overset{*}{\preccurlyeq} y$ if and only if $x_i \leq y_i$ for $i = 1,2,\ldots,k$, with $x \overset{*}{\prec} y$ if at least one of these inequalities is strict. An arbitrary total ordering $(\preccurlyeq)$ on $\mathfrak{X}$ is consistent with $(\overset{*}{\preccurlyeq})$ if $x \overset{*}{\preccurlyeq} y$ implies $x \preccurlyeq y$.

Proposition P5. If the ordering on $\mathfrak{X}$ is consistent with the natural partial ordering and Assumption A5 is satisfied, then for each $x \in \mathfrak{X}$, $P_\lambda\{X \preccurlyeq x\}$ is non-increasing in each component of λ.

Proof: For $y \in \mathfrak{X}$ we introduce the representation $y = (y_1, y^{(2)})$ where $y^{(2)} = (y_2, y_3, \ldots, y_k)$. For real z, fixed $x \in \mathfrak{X}$, and all $y \in \mathfrak{X}$, let $I_x(z, y^{(2)})$ be the indicator function of the set $\{y^{(2)} : (z, y^{(2)}) \preccurlyeq x\}$.

Then

$$P_\lambda\{X \preccurlyeq x\} = E_\lambda \, I_x(X_1, X^{(2)}) \quad . \tag{2.6}$$

For any $y \in \mathfrak{X}$, if $z' \leq z''$, then $(z', y^{(2)}) \overset{*}{\preccurlyeq} (z'', y^{(2)})$, by the consistency hypothesis, and $I_x(z', y^{(2)}) \geq I_x(z'', y^{(2)})$. Hence letting $G_x(z) = E_\lambda\{I_x(X_1, X^{(2)}) | X_1 = z\}$ we see that $G_x(z)$ is non-increasing in z. Thus, since the distribution of X_1 is stochastically increasing in λ_1 (A5), we conclude that $E_\lambda \, I_x(X_1, X^{(2)}) = E_\lambda \, G_x(X_1)$ is non-increasing in λ_1. The same argument applies to the other components of λ establishing the desired result. $\square$

Suppose that Λ is the non-negative orthant of $R^{(k)}$ and let $\mathcal{S}$ be the simplex $\mathcal{S} = \{\lambda : \Sigma_{i=1}^{k} \lambda_i = 1\}$. If A4 and A5 are satisfied, then by P5 we observe that for any $\lambda \in \mathcal{S}$ and real c, $P_{c\lambda}\{X \preccurlyeq x\}$ is continuous and non-increasing in c. If the lower bound of $P_{c\lambda}\{X \preccurlyeq x\}$ as $c \to \infty$ is less than α for all $x \in \mathfrak{X}$ and all $\lambda \in \mathcal{S}$, then for each $x \in \mathfrak{X}$ and $\lambda \in \mathcal{S}$ there exists a smallest number $b = b(x,\lambda)$ such that $P_{b(x,\lambda)\lambda}\{X \preccurlyeq x\} = \alpha$. The confidence bound $t(x)$ defined by D2 is then given by

$$t(x) = \sup_{\lambda \in \mathcal{S}} \theta(b(x,\lambda)\lambda) \quad . \tag{2.7}$$

Now $b(x,\lambda)$ is easily computed using root-finding techniques so that the computation of $t(x)$ reduces to searching over $\mathcal{S}$ for the maximum of $\theta(b(x,\lambda)\lambda)$. Many routines are available for implementing such searches. For the situation described in Corollary C1, the value of

b such that $P_{b\lambda}\{X \preccurlyeq x\} = \alpha$ is unique and (2.7) is a computationally feasible version of (1.5).

All of the above results apply mutatis mutandis to the construction of lower confidence bounds and hence confidence intervals. Applications to the reliability of coherent systems involving the binomial or other distributions are possible. In particular, the above discussion applies directly to the binomial case under the transformation $\lambda_i = -\log(1-p_i)$, $i = 1,2,\ldots,k$; $\theta(\lambda) = \Sigma_{i=1}^{k} \lambda_i = -\log \Pi_{i=1}^{k} (1-p_i)$.

3. Systems With k = 2

As was noted in Section 1, if the system has effectively only one component (e.g., when all sample sizes are equal), then the problem reduces to the well-known case of finding an upper confidence bound for a single Poisson parameter. Then in the notation of Section 1, $\theta(\lambda) = \lambda$ and if $t(x)$ is the confidence bound for λ, the lower confidence bound (1.6) for reliability R becomes

$$r(x) = 1 - t(x)/n \quad , \tag{3.1}$$

where n is the (common) sample size.

The two component case (k = 2) presents all of the difficulties of the general case. The principal problem is to generate an ordering of the sample points $x = (x_1,x_2)$ which will lead to a "good" confidence bound $t(x)$ computed using (1.5) or (2.7). Several different methods have been considered and implemented to varying extents

in the course of this investigation. These methods may be described briefly as follows:

(i) The x's are ordered according to the values of the function $\tilde{t}(x) = a_1x_1 + a_2x_2 + z_\alpha\sqrt{a_1^2x_1 + a_2^2x_2}$, where z_α is the upper α-th quantile of the standard normal distribution.

(ii) The x's are ordered according to the values of the approximate confidence bound obtained from the maximum likelihood ratio statistic (see Section 4).

(iii) The ordering is generated sequentially by considering at each stage the group of points which are not yet ordered but could be adjoined without violating the natural partial ordering (see D3 of Section 2). The next point in the ordering is then chosen to be the "best" member of the candidate group, i.e., the point producing the smallest value of t(x) given by (1.5).

(iv) The ordering is chosen so as to minimize $E_G\{t(X)\}$ for some suitable prior distribution G over the values of λ.

(v) The ordering is generated sequentially in the manner of (iii) above except that at each stage the candidate points for the next <u>two</u> steps are considered as pairs and the next point selected is the first step component which, together with the best available point for the second step, produces the smallest sum for the two values of t(x). Note that the point that appears to be "best" two steps ahead may not actually be chosen when that stage is reached.

It is clear that none of these methods is special to the case $k = 2$. Method (i), based on the function $\tilde{t}(x)$, which is really a maximum likelihood estimate of an asymptotically valid confidence bound, was used to generate tables of bounds for the case $k = 2$ in Johns (1975). Method (ii) does not improve substantially on Method (i) for moderate values of the X_i's. Methods (i) and (ii) are asymptotically equivalent when at least one X_i becomes large (see Johns 1975) and indeed standard maximum likelihood results guarantee that both are asymptotically optimal. Method (iii) discussed in Johns (1977) was found to be a substantial improvement on (i) in the strong sense that when the Method (iii) ordering is used the values of $t(x)$ are often smaller and only rarely slightly larger than the values for corresponding x's produced by Method (i).

The semi-Bayesian approach of Method (iv), which minimizes the expected length of the confidence interval, is the only one of the five that is directly motivated by optimality considerations. The bound resulting from any reasonable prior must at least be admissible. In pursuing this approach it was decided in the spirit of objectivity and in the hope of rapid convergence to asymptotic optimality to choose a prior distribution leading to an unconditional probability mass function for x_1 and x_2 constant for constant values of the maximum likelihood estimator $a_1x_1 + a_2x_2$ for $\theta(\lambda)$. In particular, the prior density for λ_1 and λ_2 was taken to be

$$g(\lambda_1,\lambda_2) = b_1b_2\, e^{-b_1\lambda_1-b_2\lambda_2},\ \lambda_1,\lambda_2 > 0 \ , \qquad (3.2)$$

where $b_1 = (1-e^{-\beta a_1})$, $b_2 = (1-e^{-\beta a_2})$, $\beta > 0$. This produces the unconditional probability mass function

$$p(x_1,x_2) = b_1 b_2 \, e^{-\beta(a_1 x_1 + a_2 x_2)}, \quad x_1,x_2 = 0,1,\ldots, \quad . \tag{3.3}$$

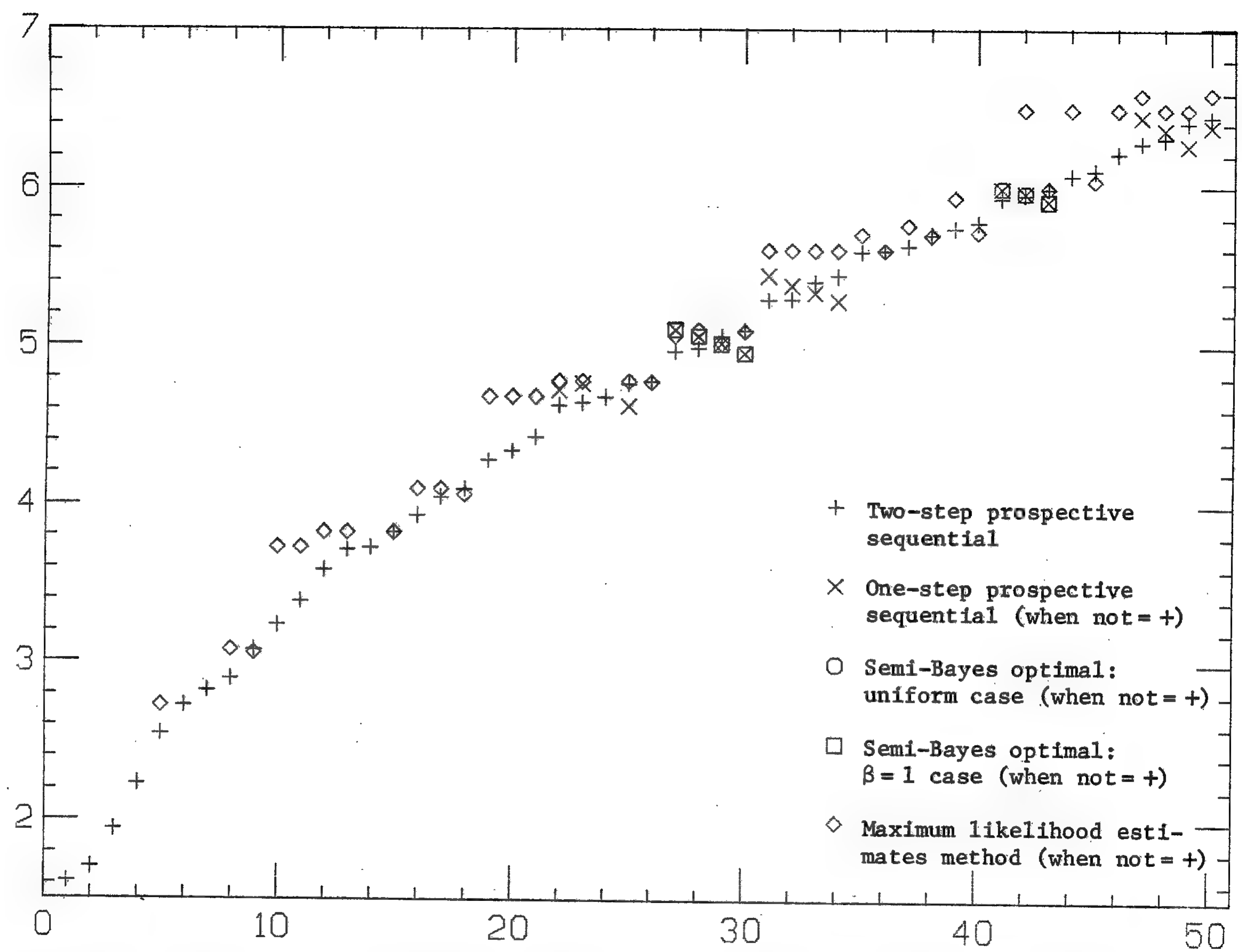
In the limiting case, as $\beta \to 0$, $p(x_1,x_2)$ becomes essentially uniform over any finite set of points (x_1,x_2).

The actual minimization of $E_G\{t(X)\}$ may, in principle, be accomplished by finding the ordering which minimizes the contribution to E_G among all orderings of length N where N may be arbitrarily large. This may be done systematically by starting at the origin (0,0) and constructing a tree whose nodes at each stage are characterized by a candidate point newly adjoined to the ordering and the corresponding value of $\Sigma p(x)t(x)$, where the sum is taken over all x's occurring in the path leading to the node, including the one just adjoined. At the N-th stage the node having the smallest accumulated sum identifies the optimal ordering of length N. This process may be facilitated by eliminating duplicate nodes and discontinuing branches when a node is reached whose value exceeds that known to be attainable in N stages. Nevertheless, because of the rapid increase in the number of nodes considered per stage, only the first forty or so points in the optimal orderings could be determined even using a very large computer facility.

In order to obtain examples of admissible orderings with which to compare the results of other methods, this computation was performed for two cases using an IBM 370/168. For both cases the

values α = .10 and a_1 = .30 were used. For the first case the probabilities given in (3.3) with β = 1 were used and the first 41 points of the optimal ordering were obtained. The forty-first stage of the computation produced 4557 nodes. For the second case the limiting situation as $\beta \to 0$ where the p(x)'s are all equal was used and the first 43 points of the optimal ordering were obtained. The number of nodes produced at the forty-third stage was 6478.

A comparison of these results with the corresponding results obtained using Methods (i), (iii), and (v) is indicated in Figure 1. The horizontal axis indexes the first 50 points in the ordering produced by Method (v), the two-step prospective sequential procedure. The values of t(x) for these indexed points for the five methods are indicated by the plotted symbols. Values of t(x) for methods other than (v) are shown only when they differ from those produced by that method. Based on this evidence it appears that Method (iv) and Method (v) differ very little and that both are better than the other methods. In fact, Method (iv) for the uniform case (β = 0) differs only trivially from Method (v). Since the use of Method (iv) for the construction of tables is now and probably always will be impractical, we are led to the choice for this purpose of the more tractible and virtually equivalent Method (v). Prospective sequential methods looking ahead more than two steps might be feasible, although the complexity of the computations increases rapidly with the number of steps. However, such procedures would be expected to produce only minute improvement over the two-step method.



<u>Figure 1</u>. Comparison of confidence bounds for five ordering methods: $\alpha = .10$, $a_1 = .30$

Table 1 gives values of the upper confidence bound $t(x)$ computed using (2.7) with the sample orderings generated by the two-step prospective sequential Method (v). The values are given for the first 100 points in each ordering for $\alpha = .01, .05, .10$, for $a_1 = .05(.05).45$. It is assumed that the components of the system are indexed so that $a_1 < a_2$ which implies $a_1 < .50$. For convenience the values of (x_1, x_2) are listed systematically rather than in the order generated by the two-step procedure. This table provides a basis for computing accurate confidence bounds for the case $k = 2$ using only simple interpolation. If values of a_1 greater than .45 but (necessarily) less than .50 are required, the bound for $a_1 = .5$ (corresponding to $n_1 = n_2$) may be used for interpolation. This bound is obtained by simply multiplying the ordinary upper confidence bound for a single Poisson parameter based on $x = x_1 + x_2$ failures by .50 (see, e.g., Pearson and Hartley 1958 for tables). The use of Table 1 is illustrated by the following two examples:

Example 1. Suppose that the two components of a series system are tested independently using sample sizes $n_1 = 300$ and $n_2 = 100$ respectively with the corresponding observed numbers of failures $X_1 = 3$ and $X_2 = 4$. Then $c = (1/300 + 1/100) = 4/300$ and $a_1 = 1/cn_1 = .25 = 1 - a_2$. If we wish to find a 95 percent confidence interval, we take $\alpha = .05$, and from Table 1 we find the confidence bound $t(x)$ for $\theta(\lambda)$ to be 7.333. Hence by (1.6) the 95 percent lower confidence bound for system reliability R is $1 - (4/300)(7.333) = .902$.

Table 1. The Confidence Bound t(x) for k = 2 for the First 100 Points Generated by the Two-Stage Optimal Ordering Method for Each a_1 and α.

a1 = 0.05			alpha = 0.01		
x1	x2	t(x1,x2)	x1	x2	t(x1,x2)
0	0	4.375	25	0	5.262
0	1	6.306	25	1	7.195
1	0	4.376	26	0	5.306
1	1	6.308	26	1	7.239
2	0	4.387	27	0	5.351
2	1	6.319	27	1	7.284
3	0	4.407	28	0	5.395
3	1	6.327	28	1	7.329
4	0	4.432	29	0	5.440
4	1	6.364	30	0	5.485
5	0	4.461	31	0	5.530
5	1	6.393	32	0	5.576
6	0	4.493	33	0	5.621
6	1	6.424	34	0	5.667
7	0	4.526	35	0	5.712
7	1	6.458	36	0	5.758
8	0	4.561	37	0	5.804
8	1	6.493	38	0	5.850
9	0	4.598	39	0	5.896
9	1	6.529	40	0	5.942
10	0	4.635	41	0	5.989
10	1	6.567	42	0	6.035
11	0	4.673	43	0	6.081
11	1	6.605	44	0	6.128
12	0	4.712	45	0	6.174
12	1	6.644	46	0	6.221
13	0	4.752	47	0	6.268
13	1	6.684	48	0	6.315
14	0	4.792	49	0	6.362
14	1	6.724	50	0	6.409
15	0	4.833	51	0	6.456
15	1	6.765	52	0	6.503
16	0	4.875	53	0	6.550
16	1	6.807	54	0	6.597
17	0	4.916	55	0	6.644
17	1	6.849	56	0	6.692
18	0	4.959	57	0	6.739
18	1	6.891	58	0	6.786
19	0	5.001	59	0	6.834
19	1	6.933	60	0	6.881
20	0	5.044	61	0	6.929
20	1	6.976	62	0	6.977
21	0	5.087	63	0	7.024
21	1	7.019	64	0	7.072
22	0	5.130	65	0	7.120
22	1	7.063	66	0	7.168
23	0	5.174	67	0	7.215
23	1	7.107	68	0	7.263
24	0	5.218	69	0	7.311
24	1	7.151	70	0	7.359

a1 = 0.05			alpha = 0.05		
x1	x2	t(x1,x2)	x1	x2	t(x1,x2)
0	0	2.846	25	0	3.733
0	1	4.507	25	1	5.396
1	0	2.847	26	0	3.777
1	1	4.508	26	1	5.440
2	0	2.858	27	0	3.822
2	1	4.519	27	1	5.485
3	0	2.878	28	0	3.866
3	1	4.539	28	1	5.529
4	0	2.903	29	0	3.911
4	1	4.564	29	1	5.574
5	0	2.932	30	0	3.956
5	1	4.593	30	1	5.619
6	0	2.964	31	0	4.001
6	1	4.624	31	1	5.665
7	0	2.997	32	0	4.047
7	1	4.658	33	0	4.092
8	0	3.032	34	0	4.138
8	1	4.693	35	0	4.183
9	0	3.069	36	0	4.229
9	1	4.730	37	0	4.275
10	0	3.106	38	0	4.321
10	1	4.767	39	0	4.367
11	0	3.144	40	0	4.413
11	1	4.805	41	0	4.460
12	0	3.183	42	0	4.506
12	1	4.845	43	0	4.552
13	0	3.223	44	0	4.599
13	1	4.884	45	0	4.645
14	0	3.264	46	0	4.692
14	1	4.925	47	0	4.739
15	0	3.304	48	0	4.786
15	1	4.966	49	0	4.833
16	0	3.346	50	0	4.880
16	1	5.007	51	0	4.927
17	0	3.388	52	0	4.974
17	1	5.049	53	0	5.021
18	0	3.430	54	0	5.068
18	1	5.091	55	0	5.115
19	0	3.472	56	0	5.163
19	1	5.134	57	0	5.210
20	0	3.515	58	0	5.257
20	1	5.177	59	0	5.305
21	0	3.558	60	0	5.353
21	1	5.220	61	0	5.400
22	0	3.601	62	0	5.448
22	1	5.264	63	0	5.495
23	0	3.645	64	0	5.543
23	1	5.307	65	0	5.591
24	0	3.689	66	0	5.639
24	1	5.351	67	0	5.686

a1 = 0.05			alpha = 0.10		
x1	x2	t(x1,x2)	x1	x2	t(x1,x2)
0	0	2.188	25	0	3.075
0	1	3.695	25	1	4.585
1	0	2.189	26	0	3.119
1	1	3.697	26	1	4.629
2	0	2.200	27	0	3.163
2	1	3.708	27	1	4.674
3	0	2.220	28	0	3.208
3	1	3.727	28	1	4.718
4	0	2.245	29	0	3.253
4	1	3.753	29	1	4.763
5	0	2.274	30	0	3.298
5	1	3.781	30	1	4.809
6	0	2.305	31	0	3.343
6	1	3.813	31	1	4.854
7	0	2.339	32	0	3.388
7	1	3.847	32	1	4.899
8	0	2.374	33	0	3.434
8	1	3.882	33	1	4.945
9	0	2.410	34	0	3.479
9	1	3.918	35	0	3.525
10	0	2.447	36	0	3.571
10	1	3.956	37	0	3.616
11	0	2.486	38	0	3.662
11	1	3.994	39	0	3.709
12	0	2.525	40	0	3.755
12	1	4.033	41	0	3.801
13	0	2.565	42	0	3.847
13	1	4.073	43	0	3.894
14	0	2.605	44	0	3.940
14	1	4.114	45	0	3.987
15	0	2.646	46	0	4.034
15	1	4.154	47	0	4.081
16	0	2.687	48	0	4.127
16	1	4.196	49	0	4.174
17	0	2.729	50	0	4.221
17	1	4.238	51	0	4.268
18	0	2.771	52	0	4.315
18	1	4.280	53	0	4.362
19	0	2.814	54	0	4.410
19	1	4.323	55	0	4.457
20	0	2.856	56	0	4.504
20	1	4.366	57	0	4.552
21	0	2.900	58	0	4.599
21	1	4.409	59	0	4.647
22	0	2.943	60	0	4.694
22	1	4.453	61	0	4.742
23	0	2.987	62	0	4.789
23	1	4.496	63	0	4.837
24	0	3.031	64	0	4.885
24	1	4.540	65	0	4.932

Table 1. (Continued)

a1 = 0.1	alpha = 0.01				
x1	x2	t(x1,x2)	x1	x2	t(x1,x2)
0	0	4.145	17	0	5.384
0	1	5.975	17	1	7.221
0	2	7.565	18	0	5.477
1	0	4.151	18	1	7.314
1	1	5.981	19	0	5.569
1	2	7.571	19	1	7.407
2	0	4.184	20	0	5.662
2	1	6.014	20	1	7.501
2	2	7.605	21	0	5.756
3	0	4.235	21	1	7.596
3	1	6.065	22	0	5.850
3	2	7.656	22	1	7.691
4	0	4.297	23	0	5.945
4	1	6.127	23	1	7.786
4	2	7.718	24	0	6.040
5	0	4.366	24	1	7.882
5	1	6.196	25	0	6.135
5	2	7.786	25	1	7.978
6	0	4.439	26	0	6.230
6	1	6.270	26	1	8.075
6	2	7.861	27	0	6.326
7	0	4.516	27	1	8.172
7	1	6.348	28	0	6.422
7	2	7.939	28	1	8.270
8	0	4.596	29	0	6.519
8	1	6.428	29	1	8.369
8	2	8.020	30	0	6.616
9	0	4.679	30	1	8.467
9	1	6.510	31	0	6.713
9	2	8.102	31	1	8.565
10	0	4.763	32	0	6.810
10	1	6.595	32	1	8.665
10	2	8.187	33	0	6.908
11	0	4.848	34	0	7.006
11	1	6.681	35	0	7.104
11	2	8.273	36	0	7.202
12	0	4.935	37	0	7.300
12	1	6.768	38	0	7.399
12	2	8.361	39	0	7.498
13	0	5.023	40	0	7.600
13	1	6.857	41	0	7.700
13	2	8.450	42	0	7.799
14	0	5.112	43	0	7.899
14	1	6.947	44	0	8.000
14	2	8.540	45	0	8.100
15	0	5.202	46	0	8.200
15	1	7.037	47	0	8.301
15	2	8.631	48	0	8.402
16	0	5.293	49	0	8.503
16	1	7.129	50	0	8.604

a1 = 0.1	alpha = 0.05				
x1	x2	t(x1,x2)	x1	x2	t(x1,x2)
0	0	2.696	16	1	5.426
0	1	4.269	16	2	6.827
0	2	5.666	17	0	3.936
0	3	6.978	17	1	5.519
1	0	2.702	17	2	6.920
1	1	4.275	18	0	4.028
1	2	5.672	18	1	5.612
2	0	2.735	19	0	4.121
2	1	4.309	19	1	5.706
2	2	5.706	20	0	4.214
3	0	2.787	20	1	5.800
3	1	4.360	21	0	4.307
3	2	5.757	21	1	5.895
4	0	2.849	22	0	4.402
4	1	4.422	22	1	5.991
4	2	5.819	23	0	4.496
5	0	2.917	23	1	6.087
5	1	4.491	24	0	4.591
5	2	5.889	24	1	6.183
6	0	2.991	25	0	4.687
6	1	4.565	25	1	6.280
6	2	5.963	26	0	4.782
7	0	3.068	26	1	6.380
7	1	4.643	27	0	4.878
7	2	6.040	27	1	6.478
8	0	3.148	28	0	4.975
8	1	4.723	28	1	6.577
8	2	6.121	29	0	5.073
9	0	3.230	29	1	6.676
9	1	4.806	30	0	5.170
9	2	6.204	30	1	6.776
10	0	3.314	31	0	5.268
10	1	4.891	31	1	6.878
10	2	6.289	32	0	5.366
11	0	3.400	33	0	5.464
11	1	4.977	34	0	5.563
11	2	6.376	35	0	5.661
12	0	3.487	36	0	5.760
12	1	5.065	37	0	5.859
12	2	6.464	38	0	5.959
13	0	3.575	39	0	6.058
13	1	5.154	40	0	6.158
13	2	6.553	41	0	6.258
14	0	3.664	42	0	6.359
14	1	5.244	43	0	6.459
14	2	6.644	44	0	6.559
15	0	3.754	45	0	6.660
15	1	5.334	46	0	6.761
15	2	6.735	47	0	6.862
16	0	3.844	48	0	6.963

a1 = 0.1	alpha = 0.10				
x1	x2	t(x1,x2)	x1	x2	t(x1,x2)
0	0	2.072	15	1	4.568
0	1	3.501	15	2	5.862
0	2	4.790	16	0	3.221
0	3	6.013	16	1	4.659
1	0	2.078	16	2	5.954
1	1	3.507	17	0	3.312
1	2	4.796	17	1	4.752
1	3	6.019	17	2	6.048
2	0	2.111	18	0	3.404
2	1	3.540	18	1	4.850
2	2	4.829	19	0	3,497
2	3	6.052	19	1	4.940
3	0	2.163	20	0	3.590
3	1	3.590	20	1	5.035
3	2	4.881	21	0	3.684
3	3	6.104	21	1	5.130
4	0	2.225	22	0	3.778
4	1	3.654	22	1	5.227
4	2	4.943	23	0	3.873
5	0	2.293	23	1	5.323
5	1	3.723	24	0	3.968
5	2	5.013	24	1	5.423
6	0	2.367	25	0	4.064
6	1	3.797	25	1	5.520
6	2	5.087	26	0	4.160
7	0	2.444	26	1	5.619
7	1	3.875	27	0	4.257
7	2	5.165	27	1	5.718
8	0	2.524	28	0	4.354
8	1	3.955	28	1	5.817
8	2	5.246	29	0	4.451
9	0	2.606	29	1	5.916
9	1	4.038	30	0	4.548
9	2	5.329	30	1	6.016
10	0	2.690	31	0	4.646
10	1	4.123	31	1	6.116
10	2	5.414	32	0	4.744
11	0	2.776	33	0	4.843
11	1	4.209	34	0	4.950
11	2	5.501	35	0	5.049
12	0	2.863	36	0	5.149
12	1	4.297	37	0	5.249
12	2	5.590	38	0	5.349
13	0	2.951	39	0	5.450
13	1	4.386	40	0	5.550
13	2	5.679	41	0	5.651
14	0	3.040	42	0	5.752
14	1	4.476	43	0	5.853
14	2	5.770	44	0	5.955
15	0	3.130	45	0	6.056

Table 1. (Continued)

a1 = 0.15			alpha = 0.01		
x1	x2	t(x1,x2)	x1	x2	t(x1,x2)
0	0	3.914	13	1	7.143
0	1	5.643	13	2	8.659
0	2	7.145	14	0	5.538
0	3	8.538	14	1	7.291
1	0	3.929	14	2	8.809
1	1	5.658	15	0	5.684
1	2	7.160	15	1	7.443
1	3	8.553	15	2	8.964
2	0	3.994	16	0	5.831
2	1	5.723	16	1	7.594
2	2	7.225	16	2	9.117
2	3	8.619	17	0	5.979
3	0	4.087	17	1	7.747
3	1	5.816	17	2	9.272
3	2	7.319	18	0	6.128
3	3	8.712	18	1	7.901
4	0	4.194	18	2	9.427
4	1	5.924	19	0	6.278
4	2	7.427	19	1	8.056
4	3	8.821	19	2	9.583
5	0	4.311	20	0	6.429
5	1	6.042	20	1	8.212
5	2	7.546	21	0	6.588
5	3	8.941	21	1	8.381
6	0	4.435	22	0	6.741
6	1	6.167	22	1	8.538
6	2	7.672	23	0	6.896
6	3	9.067	23	1	8.696
7	0	4.564	24	0	7.050
7	1	6.297	24	1	8.855
7	2	7.803	25	0	7.206
7	3	9.199	25	1	9.014
8	0	4.696	26	0	7.362
8	1	6.435	26	1	9.173
8	2	7.939	27	0	7.519
8	3	9.336	27	1	9.333
9	0	4.831	28	0	7.675
9	1	6.569	28	1	9.492
9	2	8.078	29	0	7.833
9	3	9.476	30	0	7.991
10	0	4.969	31	0	8.148
10	1	6.709	32	0	8.307
10	2	8.223	33	0	8.465
11	0	5.109	34	0	8.624
11	1	6.852	35	0	8.783
11	2	8.364	36	0	8.942
12	0	5.251	37	0	9.101
12	1	6.997	38	0	9.260
12	2	8.510	39	0	9.420
13	0	5.394	40	0	9.580

a1 = 0.15			alpha = 0.05		
x1	x2	t(x1,x2)	x1	x2	t(x1,x2)
0	0	2.546	12	0	3.883
0	1	4.032	12	1	5.403
0	2	5.351	12	2	6.728
0	3	6.591	13	0	4.026
0	4	7.780	13	1	5.560
1	0	2.561	13	2	6.881
1	1	4.047	14	0	4.171
1	2	5.367	14	1	5.717
1	3	6.606	14	2	7.034
1	4	7.796	15	0	4.319
2	0	2.626	15	1	5.874
2	1	4.113	15	2	7.187
2	2	5.432	16	0	4.468
2	3	6.671	16	1	6.032
3	0	2.719	16	2	7.341
3	1	4.206	17	0	4.617
3	2	5.525	17	1	6.190
3	3	6.765	17	2	7.496
4	0	2.826	18	0	4.768
4	1	4.314	18	1	6.348
4	2	5.635	18	2	7.651
4	3	6.875	19	0	4.919
5	0	2.943	19	1	6.506
5	1	4.433	19	2	7.807
5	2	5.754	20	0	5.072
5	3	6.995	20	1	6.665
6	0	3.067	21	0	5.225
6	1	4.559	21	1	6.824
6	2	5.881	22	0	5.378
6	3	7.123	22	1	6.982
7	0	3.196	23	0	5.532
7	1	4.690	23	1	7.141
7	2	6.013	24	0	5.687
7	3	7.256	24	1	7.300
8	0	3.328	25	0	5.843
8	1	4.825	25	1	7.459
8	2	6.150	26	0	5.998
8	3	7.394	26	1	7.617
9	0	3.463	27	0	6.154
9	1	4.964	27	1	7.776
9	2	6.290	28	0	6.310
9	3	7.536	29	0	6.467
10	0	3.601	30	0	6.624
10	1	5.105	31	0	6.781
10	2	6.433	32	0	6.938
10	3	7.681	33	0	7.096
11	0	3.741	34	0	7.253
11	1	5.249	35	0	7.411
11	2	6.579	36	0	7.569
11	3	7.829	37	0	7.727

a1 = 0.15			alpha = 0.10		
x1	x2	t(x1,x2)	x1	x2	t(x1,x2)
0	0	1.957	11	2	5.755
0	1	3.306	11	3	6.921
0	2	4.524	12	0	3.293
0	3	5.679	12	1	4.679
0	4	6.795	12	2	5.905
1	0	1.972	13	0	3.438
1	1	3.321	13	1	4.836
1	2	4.530	13	2	6.055
1	3	5.694	14	0	3.583
1	4	6.810	14	1	4.993
2	0	2.037	14	2	6.206
2	1	3.387	15	0	3.735
2	2	4.605	15	1	5.150
2	3	5.759	15	2	6.368
2	4	6.875	16	0	3.884
3	0	2.129	16	1	5.307
3	1	3.480	16	2	6.522
3	2	4.698	17	0	4.035
3	3	5.854	17	1	5.464
3	4	6.970	17	2	6.676
4	0	2.237	18	0	4.186
4	1	3.589	18	1	5.623
4	2	4.808	18	2	6.830
4	3	5.964	19	0	4.339
5	0	2.354	19	1	5.780
5	1	3.708	19	2	6.984
5	2	4.928	20	0	4.491
5	3	6.085	20	1	5.938
6	0	2.478	21	0	4.645
6	1	3.834	21	1	6.096
6	2	5.056	22	0	4.799
6	3	6.216	22	1	6.254
7	0	2.606	23	0	4.954
7	1	3.966	23	1	6.411
7	2	5.189	24	0	5.109
7	3	6.347	24	1	6.569
8	0	2.739	25	0	5.264
8	1	4.102	25	1	6.727
8	2	5.327	26	0	5.419
8	3	6.485	26	1	6.885
9	0	2.874	27	0	5.575
9	1	4.241	28	0	5.731
9	2	5.469	29	0	5.887
9	3	6.627	30	0	6.044
10	0	3.012	31	0	6.201
10	1	4.384	32	0	6.357
10	2	5.610	33	0	6.514
10	3	6.772	34	0	6.671
11	0	3.152	35	0	6.828
11	1	4.529	36	0	6.986

Table 1. (Continued)

a1 = 0.2			alpha = 0.01			a1 = 0.2			alpha = 0.05			a1 = 0.2			alpha = 0.10		
x1	x2	t(x1,x2)	x1	x2	t(x1,x2)	x1	x2	t(x1,x2)	x1	x2	t(x1,x2)	x1	x2	t(x1,x2)	x1	x2	t(x1,x2)
0	0	3.684	10	2	8.377	0	0	2.397	9	3	7.664	0	0	1.842	9	1	4.572
0	1	5.311	10	3	9.716	0	1	3.795	10	0	3.982	0	1	3.112	9	2	5.716
0	2	6.725	11	0	5.469	0	2	5.037	10	1	5.440	0	2	4.258	9	3	6.820
0	3	8.036	11	1	7.153	0	3	6.203	10	2	6.697	0	3	5.345	10	0	3.438
0	4	9.284	11	2	8.590	0	4	7.323	10	3	7.873	0	4	6.395	10	1	4.786
1	0	3.714	11	3	9.938	0	5	8.410	11	0	4.187	0	5	7.420	10	2	5.925
1	1	5.341	12	0	5.676	1	0	2.427	11	1	5.652	1	0	1.872	10	3	7.030
1	2	6.755	12	1	7.381	1	1	3.825	11	2	6.908	1	1	3.142	11	0	3.646
1	3	8.067	12	2	8.805	1	2	5.067	11	3	8.084	1	2	4.288	11	1	5.001
1	4	9.314	12	3	10.155	1	3	6.233	12	0	4.395	1	3	5.375	11	2	6.135
2	0	3.821	13	0	5.885	1	4	7.353	12	1	5.865	1	4	6.425	11	3	7.261
2	1	5.449	13	1	7.598	1	5	8.441	12	2	7.120	1	5	7.450	12	0	3.856
2	2	6.864	13	2	9.020	2	0	2.534	12	3	8.295	2	0	1.979	12	1	5.215
2	3	8.176	14	0	6.096	2	1	3.934	13	0	4.605	2	1	3.251	12	2	6.345
2	4	9.424	14	1	7.815	2	2	5.176	13	1	6.158	2	2	4.398	12	3	7.471
3	0	3.965	14	2	9.236	2	3	6.343	13	2	7.332	2	3	5.486	13	0	4.067
3	1	5.595	15	0	6.308	2	4	7.464	13	3	8.506	2	4	6.536	13	1	5.430
3	2	7.011	15	1	8.034	2	5	8.552	14	0	4.816	2	5	7.561	13	2	6.555
3	3	8.324	15	2	9.452	3	0	2.677	14	1	6.320	3	0	2.123	13	3	7.681
3	4	9.573	16	0	6.522	3	1	4.081	14	2	7.544	3	1	3.399	14	0	4.279
4	0	4.128	16	1	8.252	3	2	5.325	15	0	5.072	3	2	4.547	14	1	5.644
4	1	5.761	16	2	9.669	3	3	6.493	15	1	6.536	3	3	5.636	14	2	6.766
4	2	7.179	17	0	6.776	3	4	7.614	15	2	7.757	3	4	6.688	15	0	4.491
4	3	8.494	17	1	8.471	4	0	2.840	16	0	5.288	3	5	7.714	15	1	5.859
4	4	9.744	17	2	9.845	4	1	4.248	16	1	6.752	4	0	2.286	15	2	6.930
5	0	4.302	18	0	6.996	4	2	5.496	16	2	7.970	4	1	3.568	16	0	4.705
5	1	5.940	18	1	8.691	4	3	6.664	17	0	5.506	4	2	4.719	16	1	6.073
5	2	7.361	18	2	10.060	4	4	7.787	17	1	6.967	4	3	5.810	16	2	7.139
5	3	8.678	19	0	7.216	5	0	3.015	17	2	8.183	4	4	6.864	17	0	4.919
5	4	9.926	19	1	8.910	5	1	4.430	18	0	5.723	5	0	2.460	17	1	6.287
6	0	4.485	19	2	10.276	5	2	5.681	18	1	7.183	5	1	3.751	17	2	7.349
6	1	6.129	20	0	7.436	5	3	6.850	18	2	8.397	5	2	4.904	18	0	5.133
6	2	7.552	20	1	9.129	5	4	7.975	19	0	5.941	5	3	5.999	18	1	6.502
6	3	8.872	21	0	7.657	6	0	3.198	19	1	7.399	5	4	7.056	18	2	7.559
6	4	10.118	21	1	9.349	6	1	4.640	20	0	6.104	6	0	2.643	19	0	5.348
7	0	4.674	22	0	7.877	6	2	5.890	20	1	7.614	6	1	3.944	19	1	6.716
7	1	6.339	22	1	9.569	6	3	7.044	21	0	6.376	6	2	5.099	20	0	5.563
7	2	7.750	23	0	8.098	6	4	8.172	21	1	7.830	6	3	6.198	20	1	6.978
7	3	9.075	23	1	9.886	7	0	3.386	22	0	6.594	6	4	7.228	21	0	5.778
7	4	10.316	24	0	8.319	7	1	4.855	22	1	8.046	7	0	2.832	21	1	7.189
8	0	4.867	24	1	10.103	7	2	6.066	23	0	6.812	7	1	4.146	22	0	5.994
8	1	6.557	25	0	8.540	7	3	7.244	23	1	8.262	7	2	5.299	22	1	7.401
8	2	7.954	26	0	8.761	7	4	8.375	24	0	7.029	7	3	6.402	23	0	6.209
8	3	9.285	27	0	8.982	8	0	3.579	24	1	8.477	7	4	7.422	23	1	7.613
9	0	5.064	28	0	9.203	8	1	5.019	25	0	7.247	8	0	3.025	24	0	6.425
9	1	6.729	29	0	9.425	8	2	6.269	26	0	7.465	8	1	4.356	25	0	6.641
9	2	8.161	30	0	9.646	8	3	7.449	27	0	7.683	8	2	5.509	26	0	6.857
9	3	9.500	31	0	9.802	9	0	3.776	28	0	7.901	8	3	6.612	27	0	7.073
10	0	5.264	32	0	10.023	9	1	5.230	29	0	8.119	8	4	7.627	28	0	7.289
10	1	6.940	33	0	10.244	9	2	6.474	30	0	8.337	9	0	3.224	29	0	7.506

Table 1. (Continued)

a1 = 0.25			alpha = 0.01		
x1	x2	t(x1,x2)	x1	x2	t(x1,x2)
0	0	3.454	8	4	10.491
0	1	4.979	9	0	5.406
0	2	6.304	9	1	7.041
0	3	7.534	9	2	8.345
0	4	8.704	9	3	9.581
0	5	9.831	9	4	10.815
1	0	3.508	10	0	5.718
1	1	5.033	10	1	7.325
1	2	6.359	10	2	8.624
1	3	7.589	10	3	9.857
1	4	8.762	11	0	6.003
1	5	9.887	11	1	7.609
2	0	3.671	11	2	8.902
2	1	5.200	11	3	10.133
2	2	6.528	12	0	6.288
2	3	7.759	12	1	7.893
2	4	8.931	12	2	9.183
2	5	10.060	12	3	10.410
3	0	3.878	13	0	6.574
3	1	5.435	13	1	8.178
3	2	6.758	13	2	9.463
3	3	7.981	13	3	10.687
3	4	9.155	14	0	6.861
3	5	10.288	14	1	8.462
4	0	4.108	14	2	9.748
4	1	5.656	15	0	7.148
4	2	6.988	15	1	8.747
4	3	8.228	15	2	10.019
4	4	9.407	16	0	7.435
4	5	10.545	16	1	9.031
5	0	4.352	16	2	10.299
5	1	5.914	17	0	7.723
5	2	7.244	17	1	9.319
5	3	8.489	17	2	10.579
5	4	9.672	18	0	8.010
5	5	10.754	18	1	9.603
6	0	4.605	18	2	10.860
6	1	6.182	19	0	8.298
6	2	7.507	19	1	9.888
6	3	8.757	20	0	8.586
6	4	9.944	20	1	10.173
7	0	4.864	21	0	8.874
7	1	6.459	21	1	10.458
7	2	7.792	22	0	9.162
7	3	9.036	22	1	10.743
7	4	10.217	23	0	9.450
8	0	5.133	24	0	9.738
8	1	6.738	25	0	10.036
8	2	8.068	26	0	10.325
8	3	9.307	27	0	10.613

a1 = 0.25			alpha = 0.05		
x1	x2	t(x1,x2)	x1	x2	t(x1,x2)
0	0	2.247	8	1	5.307
0	1	3.558	8	2	6.432
0	2	4.722	8	3	7.527
0	3	5.815	8	4	8.680
0	4	6.865	9	0	4.210
0	5	7.885	9	1	5.686
0	6	8.882	9	2	6.809
1	0	2.301	9	3	7.907
1	1	3.613	9	4	8.950
1	2	4.777	10	0	4.486
1	3	5.871	10	1	5.856
1	4	6.921	10	2	7.087
1	5	7.941	10	3	8.181
1	6	8.938	11	0	4.830
2	0	2.464	11	1	6.243
2	1	3.781	11	2	7.360
2	2	4.948	11	3	8.455
2	3	6.044	12	0	5.111
2	4	7.096	12	1	6.524
2	5	8.118	12	2	7.633
3	0	2.671	12	3	8.594
3	1	4.001	13	0	5.392
3	2	5.172	13	1	6.684
3	3	6.274	13	2	7.781
3	4	7.333	13	3	8.863
3	5	8.368	14	0	5.607
4	0	2.901	14	1	6.961
4	1	4.267	14	2	8.058
4	2	5.422	15	0	5.954
4	3	6.537	15	1	7.237
4	4	7.599	15	2	8.331
4	5	8.569	16	0	6.174
5	0	3.145	16	1	7.514
5	1	4.549	16	2	8.729
5	2	5.639	17	0	6.458
5	3	6.738	17	1	7.869
5	4	7.793	17	2	9.003
5	5	8.815	18	0	6.798
6	0	3.398	18	1	8.143
6	1	4.755	19	0	7.079
6	2	5.961	19	1	8.418
6	3	7.000	20	0	7.369
6	4	8.052	20	1	8.628
6	5	9.075	21	0	7.650
7	0	3.665	21	1	8.906
7	1	5.035	22	0	7.932
7	2	6.166	23	0	8.213
7	3	7.262	24	0	8.495
7	4	8.312	25	0	8.777
8	0	3.936	26	0	9.058

a1 = 0.25			alpha = 0.10		
x1	x2	t(x1,x2)	x1	x2	t(x1,x2)
0	0	1.727	8	0	3.441
0	1	2.917	8	1	4.715
0	2	3.992	8	2	5.756
0	3	5.011	8	3	6.827
0	4	5.995	8	4	7.741
0	5	6.956	9	0	3.716
0	6	7.899	9	1	4.986
1	0	1.781	9	2	6.022
1	1	2.972	9	3	7.028
1	2	4.047	9	4	8.005
1	3	5.066	10	0	3.991
1	4	6.051	10	1	5.261
1	5	7.012	10	2	6.292
1	6	7.955	10	3	7.294
2	0	1.944	11	0	4.268
2	1	3.142	11	1	5.533
2	2	4.221	11	2	6.490
2	3	5.242	11	3	7.559
2	4	6.233	12	0	4.545
2	5	7.186	12	1	5.805
2	6	8.130	12	2	6.756
3	0	2.151	12	3	7.825
3	1	3.362	13	0	4.823
3	2	4.450	13	1	6.077
3	3	5.489	13	2	7.095
3	4	6.444	13	3	8.091
3	5	7.407	14	0	5.101
4	0	2.381	14	1	6.350
4	1	3.607	14	2	7.363
4	2	4.668	14	3	8.230
4	3	5.692	15	0	5.379
4	4	6.681	15	1	6.622
4	5	7.646	15	2	7.712
5	0	2.625	16	0	5.658
5	1	3.865	16	1	6.895
5	2	4.913	16	2	7.952
5	3	5.938	17	0	5.937
5	4	6.928	17	1	7.167
5	5	7.894	18	0	6.216
6	0	2.878	18	1	7.440
6	1	4.134	19	0	6.498
6	2	5.173	19	1	7.590
6	3	6.197	20	0	6.777
6	4	7.215	20	1	7.862
6	5	8.175	21	0	7.055
7	0	3.168	21	1	8.133
7	1	4.403	22	0	7.336
7	2	5.434	23	0	7.682
7	3	6.559	24	0	7.985
7	4	7.478	25	0	8.232

Table 1. (Continued)

a1 = 0.3		alpha = 0.01				a1 = 0.3		alpha = 0.05				a1 = 0.3		alpha = 0.10			
x1	x2	t(x1,x2)	x1	x2	t(x1,x2)	x1	x2	t(x1,x2)	x1	x2	t(x1,x2)	x1	x2	t(x1,x2)	x1	x2	t(x1,x2)
0	0	3.224	7	2	7.907	0	0	2.097	7	1	5.319	0	0	1.612	6	5	8.102
0	1	4.647	7	3	9.143	0	1	3.321	7	2	6.465	0	1	2.723	7	0	3.581
0	2	5.884	7	4	10.206	0	2	4.407	7	3	7.427	0	2	3.726	7	1	4.761
0	3	7.032	7	5	11.097	0	3	5.428	7	4	8.440	0	3	4.676	7	2	5.773
0	4	8.123	8	0	5.549	0	4	6.407	7	5	9.386	0	4	5.597	7	3	6.639
0	5	9.176	8	1	7.104	0	5	7.359	8	0	4.446	0	5	6.492	7	4	7.540
0	6	10.199	8	2	8.257	0	6	8.290	8	1	5.751	0	6	7.373	7	5	8.260
0	7	11.200	8	3	9.476	0	7	9.204	8	2	6.810	0	7	8.240	8	0	3.929
1	0	3.315	8	4	10.545	1	0	2.189	8	3	7.667	1	0	1.703	8	1	5.101
1	1	4.741	9	0	5.910	1	1	3.416	8	4	8.620	1	1	2.818	8	2	6.109
1	2	5.979	9	1	7.465	1	2	4.504	9	0	4.801	1	2	3.823	8	3	6.965
1	3	7.128	9	2	8.607	1	3	5.525	9	1	6.006	1	3	4.776	8	4	7.862
1	4	8.220	9	3	9.816	1	4	6.506	9	2	6.987	1	4	5.696	9	0	4.279
1	5	9.274	9	4	10.885	1	5	7.459	9	3	8.164	1	5	6.594	9	1	5.284
1	6	10.298	10	0	6.273	1	6	8.390	9	4	8.953	1	6	7.475	9	2	6.413
2	0	3.552	10	1	7.727	1	7	9.305	10	0	5.155	1	7	8.337	9	3	7.296
2	1	4.992	10	2	8.959	2	0	2.425	10	1	6.354	2	0	1.940	9	4	8.361
2	2	6.232	10	3	9.989	2	1	3.666	10	2	7.487	2	1	3.074	10	0	4.629
2	3	7.386	10	4	11.248	2	2	4.765	10	3	8.341	2	2	4.096	10	1	5.625
2	4	8.484	11	0	6.637	2	3	5.798	10	4	9.284	2	3	5.060	10	2	6.745
2	5	9.543	11	1	8.081	2	4	6.785	11	0	5.510	2	4	5.932	10	3	7.612
2	6	10.573	11	2	9.311	2	5	7.695	11	1	6.699	2	5	6.830	10	4	8.508
3	0	3.840	11	3	10.332	2	6	8.627	11	2	7.825	2	6	7.711	11	0	4.982
3	1	5.310	12	0	7.001	3	0	2.714	11	3	8.772	2	7	8.579	11	1	5.950
3	2	6.538	12	1	8.435	3	1	3.970	12	0	5.865	3	0	2.228	11	2	7.076
3	3	7.699	12	2	9.665	3	2	5.082	12	1	7.108	3	1	3.384	11	3	7.937
3	4	8.804	12	3	10.675	3	3	6.044	12	2	7.999	3	2	4.340	11	4	8.672
3	5	9.870	13	0	7.367	3	4	7.026	12	3	9.104	3	3	5.294	12	0	5.399
3	6	10.909	13	1	8.788	3	5	8.065	13	0	6.221	3	4	6.212	12	1	6.283
4	0	4.154	13	2	10.089	3	6	8.907	13	1	7.289	3	5	7.111	12	2	7.407
4	1	5.663	13	3	11.018	4	0	3.027	13	2	8.502	3	6	7.992	12	3	8.184
4	2	6.865	14	0	7.826	4	1	4.294	13	3	9.436	4	0	2.542	13	0	5.739
4	3	8.028	14	1	9.133	4	2	5.412	14	0	6.577	4	1	3.710	13	1	6.617
4	4	9.042	14	2	10.438	4	3	6.341	14	1	7.629	4	2	4.641	13	2	7.737
4	5	10.097	15	0	8.186	4	4	7.322	14	2	8.841	4	3	5.590	14	0	6.079
4	6	11.212	15	1	9.496	4	5	8.275	15	0	6.934	4	4	6.517	14	1	6.951
5	0	4.483	15	2	10.785	4	6	9.213	15	1	8.131	4	5	7.415	14	2	8.068
5	1	6.024	16	0	8.547	5	0	3.360	15	2	9.179	4	6	8.425	15	0	6.446
5	2	7.214	16	1	9.850	5	1	4.637	16	0	7.387	5	0	2.892	15	1	7.284
5	3	8.371	16	2	11.131	5	2	5.658	16	1	8.310	5	1	4.047	15	2	8.399
5	4	9.382	17	0	8.909	5	3	6.670	16	2	9.518	5	2	4.962	16	0	6.782
5	5	10.433	17	1	10.203	5	4	7.777	17	0	7.735	5	3	5.992	16	1	7.630
6	0	4.837	18	0	9.270	5	5	8.728	17	1	8.652	5	4	6.898	16	2	8.573
6	1	6.384	18	1	10.557	6	0	3.742	18	0	8.008	5	5	7.787	17	0	7.118
6	2	7.559	19	0	9.631	6	1	4.977	18	1	8.993	5	6	8.661	17	1	7.963
6	3	8.712	19	1	10.911	6	2	6.129	19	0	8.426	6	0	3.235	18	0	7.455
6	4	9.714	20	0	9.994	6	3	7.148	19	1	9.333	6	1	4.427	18	1	8.296
6	5	10.763	21	0	10.356	6	4	7.969	20	0	8.770	6	2	5.437	19	0	7.791
7	0	5.192	22	0	10.718	6	5	9.056	21	0	9.117	6	3	6.311	20	0	8.128
7	1	6.744	23	0	11.080	7	0	4.094	22	0	9.462	6	4	7.215	21	0	8.465

Table 1. (Continued)

a1 = 0.35			alpha = 0.01		
x1	x2	t(x1,x2)	x1	x2	t(x1,x2)
0	0	2.993	6	2	7.650
0	1	4.315	6	3	8.678
0	2	5.464	6	4	9.681
0	3	6.529	6	5	10.649
0	4	7.543	6	6	11.583
0	5	8.520	7	0	5.763
0	6	9.471	7	1	6.965
0	7	10.400	7	2	8.082
0	8	11.312	7	3	9.102
1	0	3.146	7	4	10.094
1	1	4.478	7	5	11.059
1	2	5.672	8	0	6.211
1	3	6.692	8	1	7.396
1	4	7.708	8	2	8.510
1	5	8.687	8	3	9.520
1	6	9.639	8	4	10.508
1	7	10.569	8	5	11.299
1	8	11.482	9	0	6.659
2	0	3.485	9	1	7.828
2	1	4.871	9	2	8.937
2	2	5.971	9	3	9.938
2	3	7.043	9	4	10.923
2	4	8.056	10	0	7.108
2	5	9.037	10	1	8.260
2	6	9.990	10	2	9.345
2	7	10.921	10	3	10.356
3	0	3.881	10	4	11.470
3	1	5.203	11	0	7.557
3	2	6.361	11	1	8.721
3	3	7.440	11	2	9.768
3	4	8.439	11	3	10.774
3	5	9.419	12	0	8.007
3	6	10.373	12	1	9.151
3	7	11.179	12	2	10.191
4	0	4.304	12	3	11.191
4	1	5.615	13	0	8.457
4	2	6.803	13	1	9.580
4	3	7.869	13	2	10.613
4	4	8.859	14	0	8.909
4	5	9.834	14	1	10.008
4	6	10.783	14	2	11.034
5	0	4.758	15	0	9.365
5	1	6.103	15	1	10.437
5	2	7.226	15	2	11.437
5	3	8.291	16	0	9.811
5	4	9.267	16	1	10.864
5	5	10.239	17	0	10.258
5	6	11.374	17	1	11.291
6	0	5.313	18	0	10.703
6	1	6.533	19	0	11.146

a1 = 0.35			alpha = 0.05		
x1	x2	t(x1,x2)	x1	x2	t(x1,x2)
0	0	1.947	6	1	5.235
0	1	3.084	6	2	6.211
0	2	4.092	6	3	7.254
0	3	5.040	6	4	7.999
0	4	5.950	6	5	8.796
0	5	6.834	7	0	4.614
0	6	7.698	7	1	5.652
0	7	8.546	7	2	6.619
0	8	9.383	7	3	7.653
1	0	2.100	7	4	8.524
1	1	3.243	7	5	9.190
1	2	4.256	8	0	5.055
1	3	5.207	8	1	5.902
1	4	6.119	8	2	7.027
1	5	7.005	8	3	7.838
1	6	7.871	8	4	8.918
1	7	8.721	8	5	9.585
1	8	9.559	9	0	5.497
2	0	2.439	9	1	6.486
2	1	3.587	9	2	7.435
2	2	4.603	9	3	8.221
2	3	5.557	9	4	9.321
2	4	6.471	10	0	6.069
2	5	7.359	10	1	6.793
2	6	8.123	10	2	8.058
2	7	8.971	10	3	8.642
3	0	2.834	10	4	9.708
3	1	3.971	11	0	6.373
3	2	4.984	11	1	7.203
3	3	5.818	11	2	8.347
3	4	6.724	11	3	9.037
3	5	7.743	12	0	6.903
3	6	8.608	12	1	7.612
3	7	9.312	12	2	8.748
4	0	3.301	12	3	9.437
4	1	4.402	13	0	7.318
4	2	5.399	13	1	7.960
4	3	6.342	13	2	9.150
4	4	7.140	13	3	9.837
4	5	8.241	14	0	7.733
4	6	9.078	14	1	8.456
4	7	9.718	14	2	9.552
5	0	3.736	15	0	8.147
5	1	4.817	15	1	8.863
5	2	5.998	16	0	8.561
5	3	6.860	16	1	9.269
5	4	7.528	17	0	8.973
5	5	8.401	17	1	9.675
5	6	9.460	18	0	9.385
6	0	4.174	19	0	9.795

a1 = 0.35			alpha = 0.10		
x1	x2	t(x1,x2)	x1	x2	t(x1,x2)
0	0	1.497	6	0	3.687
0	1	2.528	6	1	4.547
0	2	3.460	6	2	5.636
0	3	4.343	6	3	6.490
0	4	5.196	6	4	7.323
0	5	6.028	6	5	7.950
0	6	6.846	6	6	8.949
0	7	7.651	7	0	4.120
0	8	8.447	7	1	4.947
1	0	1.649	7	2	5.962
1	1	2.690	7	3	6.946
1	2	3.626	7	4	7.761
1	3	4.513	7	5	8.333
1	4	5.369	8	0	4.641
1	5	6.204	8	1	5.348
1	6	7.024	8	2	6.343
1	7	7.831	8	3	7.071
1	8	8.628	8	4	8.096
2	0	1.988	8	5	8.718
2	1	3.035	9	0	5.048
2	2	3.975	9	1	5.749
2	3	4.865	9	2	6.749
2	4	5.724	9	3	7.462
2	5	6.562	9	4	8.483
2	6	7.383	10	0	5.454
2	7	8.071	10	1	6.149
2	8	8.991	10	2	7.210
3	0	2.384	10	3	7.852
3	1	3.417	10	4	8.870
3	2	4.233	11	0	5.860
3	3	5.111	11	1	6.548
3	4	5.944	11	2	7.603
3	5	6.897	11	3	8.243
3	6	7.711	12	0	6.266
3	7	8.412	12	1	6.924
4	0	2.826	12	2	7.996
4	1	3.830	12	3	8.633
4	2	4.755	13	0	6.671
4	3	5.518	13	1	7.344
4	4	6.363	13	2	8.389
4	5	7.187	14	0	7.078
4	6	8.192	14	1	7.736
4	7	8.802	14	2	8.781
5	0	3.254	15	0	7.480
5	1	4.362	15	1	8.137
5	2	5.248	16	0	7.881
5	3	6.105	16	1	8.532
5	4	6.723	17	0	8.282
5	5	7.565	17	1	8.927
5	6	8.569	18	0	8.682

Table 1. (Continued)

a1 = 0.4		alpha = 0.01			
x1	x2	t(x1,x2)	x1	x2	t(x1,x2)
0	0	2.763	5	4	9.059
0	1	3.983	5	5	9.926
0	2	5.044	5	6	10.889
0	3	6.027	6	0	5.902
0	4	6.963	6	1	6.748
0	5	7.865	6	2	7.943
0	6	8.742	6	3	8.651
0	7	9.600	6	4	9.763
0	8	10.442	6	5	10.521
0	9	11.270	6	6	11.370
1	0	3.022	7	0	6.442
1	1	4.270	7	1	7.258
1	2	5.368	7	2	8.456
1	3	6.258	7	3	9.370
1	4	7.190	7	4	10.260
1	5	8.090	7	5	11.006
1	6	8.966	7	6	11.703
1	7	9.822	8	0	6.980
1	8	10.663	8	1	7.797
1	9	11.491	8	2	8.967
2	0	3.514	8	3	9.646
2	1	4.784	8	4	10.635
2	2	5.843	8	5	11.490
2	3	6.675	9	0	7.519
2	4	7.604	9	1	8.317
2	5	8.502	9	2	9.248
2	6	9.376	9	3	10.145
2	7	10.227	9	4	11.126
2	8	11.072	10	0	8.058
3	0	4.077	10	1	8.835
3	1	5.169	10	2	9.986
3	2	6.394	10	3	10.751
3	3	7.287	10	4	11.784
3	4	8.193	11	0	8.596
3	5	9.169	11	1	9.476
3	6	10.038	11	2	10.370
3	7	10.780	11	3	11.248
3	8	11.611	12	0	9.128
4	0	4.655	12	1	9.863
4	1	5.687	12	2	10.868
4	2	6.909	12	3	11.634
4	3	7.760	13	0	9.656
4	4	8.773	13	1	10.489
4	5	9.539	13	2	11.365
4	6	10.397	14	0	10.178
4	7	11.239	14	1	10.992
5	0	5.325	15	0	10.697
5	1	6.210	15	1	11.497
5	2	7.428	16	0	11.212
5	3	8.277	17	0	11.724

a1 = 0.4		alpha = 0.05			
x1	x2	t(x1,x2)	x1	x2	t(x1,x2)
0	0	1.797	5	3	6.770
0	1	2.846	5	4	7.580
0	2	3.778	5	5	8.373
0	3	4.652	5	6	9.153
0	4	5.492	5	7	9.913
0	5	6.308	6	0	4.737
0	6	7.114	6	1	5.576
0	7	7.889	6	2	6.421
0	8	8.661	6	3	7.241
0	9	9.423	6	4	8.047
1	0	2.057	6	5	8.836
1	1	3.080	6	6	9.577
1	2	4.103	7	0	5.259
1	3	4.876	7	1	6.065
1	4	5.713	7	2	6.899
1	5	6.528	7	3	7.712
1	6	7.324	7	4	8.523
1	7	8.107	7	5	9.301
1	8	8.878	8	0	5.774
1	9	9.640	8	1	6.552
2	0	2.549	8	2	7.377
2	1	3.503	8	3	8.183
2	2	4.415	8	4	8.983
2	3	5.282	8	5	9.751
2	4	6.118	9	0	6.282
2	5	6.932	9	1	7.037
2	6	7.728	9	2	7.854
2	7	8.502	9	3	8.653
2	8	9.272	9	4	9.615
2	9	10.033	10	0	6.785
3	0	3.158	10	1	7.521
3	1	3.982	10	2	8.330
3	2	5.086	10	3	9.123
3	3	5.831	10	4	9.985
3	4	6.651	11	0	7.283
3	5	7.452	11	1	8.002
3	6	8.238	11	2	8.804
3	7	9.013	11	3	9.397
3	8	9.778	12	0	7.777
4	0	3.679	12	1	8.482
4	1	4.581	12	2	9.524
4	2	5.450	12	3	9.868
4	3	6.289	13	0	8.267
4	4	7.104	13	1	8.960
4	5	7.911	13	2	10.059
4	6	8.694	14	0	8.755
4	7	9.461	14	1	9.229
5	0	4.209	15	0	9.357
5	1	4.971	15	1	9.703
5	2	5.942	16	0	9.833

a1 = 0.4		alpha = 0.10			
x1	x2	t(x1,x2)	x1	x2	t(x1,x2)
0	0	1.382	5	3	6.053
0	1	2.334	5	4	6.792
0	2	3.193	5	5	7.349
0	3	4.008	5	6	8.414
0	4	4.796	5	7	8.821
0	5	5.565	6	0	4.213
0	6	6.319	6	1	4.951
0	7	7.063	6	2	5.739
0	8	7.797	6	3	6.305
0	9	8.524	6	4	7.564
1	0	1.641	6	5	8.016
1	1	2.563	6	6	9.033
1	2	3.416	7	0	4.708
1	3	4.228	7	1	5.427
1	4	5.014	7	2	6.446
1	5	5.782	7	3	6.973
1	6	6.535	7	4	7.796
1	7	7.277	7	5	8.251
1	8	8.011	8	0	5.198
1	9	8.737	8	1	5.891
2	0	2.133	8	2	6.833
2	1	2.974	8	3	7.216
2	2	3.818	8	4	8.467
2	3	4.626	8	5	8.702
2	4	5.409	9	0	5.682
2	5	6.175	9	1	6.148
2	6	6.927	9	2	7.386
2	7	7.668	9	3	7.674
2	8	8.358	9	4	8.878
2	9	9.127	9	5	9.153
3	0	2.681	10	0	6.282
3	1	3.532	10	1	6.615
3	2	4.352	10	2	7.889
3	3	5.144	10	3	8.131
3	4	5.917	11	0	6.752
3	5	6.672	11	1	7.080
3	6	7.485	11	2	8.301
3	7	8.154	11	3	8.587
3	8	8.948	12	0	7.236
4	0	3.197	12	1	7.539
4	1	3.996	12	2	8.758
4	2	4.814	12	3	9.042
4	3	5.598	13	0	7.701
4	4	6.510	13	1	8.005
4	5	7.117	13	2	9.213
4	6	7.865	14	0	8.163
4	7	8.593	14	1	8.460
5	0	3.710	15	0	8.624
5	1	4.478	15	1	8.925
5	2	5.276	16	0	9.084

Table 1. (Continued)

a1 = 0.45			alpha = 0.01		
x1	x2	t(x1,x2)	x1	x2	t(x1,x2)
0	0	2.533	4	7	10.825
0	1	3.651	4	8	11.574
0	2	4.623	5	0	5.899
0	3	5.525	5	1	6.666
0	4	6.382	5	2	7.491
0	5	7.217	5	3	8.300
0	6	8.014	5	4	9.107
0	7	8.800	5	5	9.870
0	8	9.571	5	6	10.634
0	9	10.331	5	7	11.387
0	10	11.092	6	0	6.557
0	11	11.833	6	1	7.129
1	0	3.021	6	2	8.099
1	1	4.073	6	3	8.894
1	2	5.020	6	4	9.693
1	3	5.908	6	5	10.444
1	4	6.757	6	6	11.180
1	5	7.578	6	7	11.929
1	6	8.377	7	0	7.288
1	7	9.160	7	1	7.803
1	8	9.928	7	2	8.721
1	9	10.685	7	3	9.506
1	10	11.432	7	4	10.286
2	0	3.783	7	5	11.016
2	1	4.725	7	6	11.773
2	2	5.633	8	0	7.916
2	3	6.496	8	1	8.410
2	4	7.141	8	2	9.312
2	5	8.145	8	3	10.090
2	6	8.936	8	4	10.866
2	7	9.712	8	5	11.613
2	8	10.476	9	0	8.530
2	9	11.021	9	1	9.011
2	10	11.759	9	2	9.902
3	0	4.520	9	3	10.672
3	1	5.375	9	4	11.443
3	2	6.247	10	0	9.137
3	3	7.334	10	1	9.607
3	4	7.918	10	2	10.490
3	5	8.700	10	3	11.251
3	6	9.480	11	0	9.737
3	7	10.246	11	1	10.197
3	8	11.229	11	2	11.074
4	0	5.222	11	3	11.826
4	1	6.028	12	0	10.331
4	2	6.878	12	1	10.783
4	3	7.703	12	2	11.654
4	4	8.518	13	0	10.921
4	5	9.293	13	1	11.365
4	6	10.065	14	0	11.506

a1 = 0.45			alpha = 0.05		
x1	x2	t(x1,x2)	x1	x2	t(x1,x2)
0	0	1.648	4	6	8.441
0	1	2.609	4	7	9.139
0	2	3.490	4	8	9.829
0	3	4.264	5	0	4.809
0	4	5.034	5	1	5.270
0	5	5.782	5	2	6.124
0	6	6.513	5	3	6.848
0	7	7.232	5	4	7.591
0	8	7.939	5	5	8.283
0	9	8.656	5	6	8.984
0	10	9.329	5	7	9.677
0	11	10.033	6	0	5.397
1	0	2.135	6	1	5.843
1	1	3.011	6	2	6.692
1	2	3.839	6	3	7.409
1	3	4.626	6	4	8.141
1	4	5.391	6	5	8.672
1	5	6.134	6	6	9.521
1	6	6.861	6	7	10.055
1	7	7.569	7	0	5.977
1	8	8.273	7	1	6.371
1	9	8.969	7	2	7.252
1	10	9.659	7	3	7.961
2	0	2.833	7	4	8.825
2	1	3.411	7	5	9.365
2	2	4.405	8	0	6.550
2	3	5.173	8	1	6.974
2	4	5.930	8	2	7.809
2	5	6.653	8	3	8.510
2	6	7.372	8	4	9.234
2	7	8.080	8	5	9.903
2	8	8.589	9	0	7.118
2	9	9.472	9	1	7.533
2	10	9.969	9	2	8.362
3	0	3.611	9	3	9.056
3	1	4.067	9	4	9.776
3	2	4.975	10	0	7.680
3	3	5.724	10	1	8.087
3	4	6.325	10	2	8.911
3	5	7.183	10	3	9.599
3	6	7.895	10	4	10.129
3	7	8.795	11	0	8.238
3	8	9.293	11	1	8.638
3	9	10.174	11	2	9.457
4	0	4.211	12	0	8.791
4	1	4.689	12	1	9.185
4	2	5.555	12	2	10.001
4	3	6.478	13	0	9.341
4	4	7.037	13	1	9.730
4	5	7.735	14	0	9.887

a1 = 0.45			alpha = 0.10		
x1	x2	t(x1,x2)	x1	x2	t(x1,x2)
0	0	1.266	4	6	7.795
0	1	2.139	4	7	8.490
0	2	2.964	4	8	9.116
0	3	3.674	5	0	4.233
0	4	4.419	5	1	4.540
0	5	5.101	5	2	5.470
0	6	5.793	5	3	6.152
0	7	6.474	5	4	6.856
0	8	7.147	5	5	7.505
0	9	7.813	5	6	8.173
0	10	8.474	5	7	8.834
0	11	9.129	6	0	4.791
1	0	1.750	6	1	5.198
1	1	2.528	6	2	6.010
1	2	3.290	6	3	6.685
1	3	4.024	6	4	7.385
1	4	4.742	6	5	8.027
1	5	5.437	6	6	8.691
1	6	6.124	7	0	5.344
1	7	6.802	7	1	5.703
1	8	7.472	7	2	6.547
1	9	8.136	7	3	7.055
1	10	8.794	7	4	7.913
2	0	2.395	7	5	8.391
2	1	2.887	7	6	9.207
2	2	3.830	8	0	5.891
2	3	4.346	8	1	6.279
2	4	5.260	8	2	7.215
2	5	5.940	8	3	7.736
2	6	6.622	8	4	8.547
2	7	7.295	8	5	8.943
2	8	7.961	9	0	6.432
2	9	8.622	9	1	6.814
2	10	9.277	9	2	7.611
3	0	3.097	9	3	8.267
3	1	3.511	9	4	9.057
3	2	4.651	10	0	6.971
3	3	5.074	10	1	7.345
3	4	5.628	10	2	8.139
3	5	6.314	10	3	8.789
3	6	6.990	11	0	7.505
3	7	7.659	11	1	7.874
3	8	8.322	11	2	8.665
3	9	8.982	11	3	9.310
4	0	3.666	12	0	8.036
4	1	4.099	12	1	8.407
4	2	4.925	12	2	9.188
4	3	5.788	13	0	8.564
4	4	6.450	13	1	8.924
4	5	7.156	14	0	9.089

A somewhat more complicated case making use of the possibility of combining component data when some sample sizes are equal, as described in Section 1, is illustrated by

Example 2. Consider a five-component series system with test data $X_1 = 0$, $X_2 = 4$, $X_3 = 2$, $X_4 = 1$, $X_4 = 3$, based on corresponding sample sizes $n_1 = n_2 = n_3 = 300$, $n_4 = n_5 = 200$. Here $c = (3/300 + 2/200) = .02$, $a_1 = a_2 = a_3 = 1/300(.02) = 1.6$, $a_4 = a_5 = 1/200(.02) = 1/4$. For the equivalent $k^* = 2$ problem we divide the two distinct values of the a_i's by their sum $c_o = 1/6 + 1/4 = 5/12$, obtaining $a_1^* = .40$, $a_2^* = .60$. The corresponding numbers of failures are $X_1^* = X_1 + X_2 + X_3 = 6$ and $X_2^* = X_4 + X_5 = 4$. For the reduced problem we consult Table 1 for $\alpha = .10$ to find the 90 percent confidence bound 7.564. This must be multiplied by c_o to obtain the bound for the original $\theta(\lambda)$. The 90 percent lower bound $r(x)$ for the system reliability given by (1.6) thus becomes $1 - cc_o t(x) = 1 - (.02)(5/12)(7.564) = .937$.

If the observed numbers of failures fall outside the limits of Table 1, the approximate methods discussed in the next section may be employed.

4. Approximations

In this section approximate confidence bounds for systems with more than two components having distinct sample sizes are developed. The use of the maximum likelihood ratio confidence bound for cases falling outside the scope of Table 1 is also discussed.

For $k > 2$ it is impractical to generate the optimal ordering and the corresponding values of the upper confidence bound for more than a few illustrative cases. Thus, some method of approximating solutions for $k > 2$ with acceptable precision is required. The approach to be followed here is to find a $k = 2$ problem which is sufficiently similar in structure to the given $k > 2$ problem so that the confidence bounds for the two problems are essentially the same except for a normalizing factor. This method may be thought of as an extension and refinement of the Lindstrom-Madden procedure (see Lloyd and Lipow 1977, and cf. Harris and Soms 1980).

The Lindstrom-Madden method first estimates the reliability by maximum likelihood and then uses the $k = 1$ confidence bound solution for the component with the smallest sample size and a fictitious number of failures determined so as to reproduce the estimated system reliability. The procedure proposed here is to estimate two quantities, the value of θ and the variance of the maximum likelihood estimate of θ, and to use these estimates to find a $k = 2$ problem based on two of the original a_i's with a pair of corresponding fictitious observation values chosen to reproduce the estimated quantities. The two a_i's are chosen to be as large as possible (corresponding to sample sizes as small as possible) subject to two constraints. The

first constraint guarantees that the resulting fictitious observations are non-negative. The second constraint requires that a_i's corresponding to zero failures in the original problem not be considered unless all but one of the components have zero failures. These considerations lead to a unique $k = 2$ problem whose solution provides a very good approximation to the solution for the original $k > 2$ problem.

The elimination of a_i's for components exhibiting zero failures is justified by the fact that the maximum likelihood ratio confidence bound (discussed later) is invariant under such transformations. That is, if the dimension k is reduced by the elimination of all a_i's corresponding to zero failures, then the value of the maximum likelihood ratio bound remains unchanged.

The choice of the two quantities whose estimates are used to determine the pair of fictitious observations is supported by analogy with Lindstrom-Madden (in the case of θ) and by the fact that the two estimates are the ingredients of the asymptotic maximum likelihood ratio confidence bound thereby insuring asymptotic optimality. The details of the approximation algorithm are as follows:

(a) First $\theta(\lambda) = \Sigma_{i=1}^{k} a_i\lambda_i$ is estimated by

$$\hat{\theta} = \sum_{i=1}^{k} a_i X_i \quad , \tag{4.1}$$

and the quantity $\mathrm{Var}(\hat{\theta}) = \Sigma_{i=1}^{k} a_i^2\lambda_i$ is estimated by

$$\hat{v} = \sum_{i=1}^{k} a_i^2 X_i \quad . \tag{4.2}$$

(b) Next if at least one component exhibits one or more failures, the pair (a_i, a_j), $i < j$, is selected so that

(i) $a_i \leq \hat{v}/\hat{\theta}$, and

(ii) $a_j \geq \hat{v}/\hat{\theta}$.

Subject to these conditions, a_i and a_j are taken to be the largest available values associated with at least one failure. If all a_i's satisfying (i) correspond to zero failures, then a_i is taken to be the largest in that group. If all the a_j's satisfying (ii) correspond to zero failures, then a_j is taken to be a_k. If all components exhibit zero failures, $\hat{v}/\hat{\theta}$ is indeterminate and a_i and a_j are taken to be a_1 and a_k respectively.[1]

(c) The pseudo-observations x_1^* and x_2^* are computed by the formulae

$$x_1^* = \frac{a_j\hat{\theta} - \hat{v}}{a_i(a_j - a_i)} , \tag{4.3}$$

$$x_2^* = \frac{\hat{v} - a_i\hat{\theta}}{a_j(a_j - a_i)} .$$

These values will be non-negative by conditions (i) and (ii) of (b), and when associated with a_i and a_j respectively they

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[1] For this case the resulting confidence bound is exact and is the same as would be obtained by multiplying a_k times the upper confidence bound for a single Poisson parameter when zero failures are observed, i.e., $t(0) = a_k \log (1/\alpha)$.

reproduce the values of $\hat{\theta}$ and $\hat{v}$ provided all other observations are replaced by zeros.

(d) The $k = 2$ problem with x_1^* and x_2^* associated with $a_1^* = a_i/(a_i + a_j)$ and $a_2^* = a_j/(a_i + a_j)$ respectively may now be treated using Table 1 to yield $t(x_1^*, x_2^*)$. Since x_1^* and x_2^* are not necessarily integers, it may be necessary to interpolate with respect to these arguments as well as the value of $a_1 = a_1^*$.

(e) The approximate upper confidence bound t^* for θ for the original $k > 2$ problem is then given by

$$t^*(x) = (a_i + a_j)t(x_1^*, x_2^*) \quad . \tag{4.4}$$

In order to check the validity of this approximation algorithm, the confidence bounds for the first 24 points in the optimal (two-stage prospective) ordering were computed for a typical $k = 3$ case. These results were obtained by a rather laborious method based on formula (2.7) and involving repeated interactive searches of the $(\lambda_1, \lambda_2, \lambda_3)$ simplex. The approximation algorithm was applied to each of these points and the comparative results are shown in Table 2. The values of a_1, a_2, and a_3 for this example were chosen to be of roughly the same magnitude but not so close that the combination of any two would be indicated.

To check the algorithm for cases farther from the origin several additional examples were considered using a somewhat different method which avoids the necessity for sequentially generating the

Table 2. The Performance of the Approximation Algorithm for the First 24 Ordered Points for a Typical Example With k = 3.

$\alpha = .05;\ a_1 = .2,\ a_2 = .3,\ a_3 = .5$

Two-Step Prospective Sequential					Approximation Algorithm	
n	x_1	x_2	x_3	t(x)	$t^*(x)$	Relative Error
1	0	0	0	1.498	1.498	+ 00.0 %
2	1	0	0	1.553	1.555	+ 00.1
3	0	1	0	1.656	1.663	+ 00.4
4	2	0	0	1.703	1.705	+ 00.2
5	1	1	0	1.756	1.540	- 12.3
6	2	1	0	1.861	1.752	- 05.9
7	3	0	0	1.933	1.891	- 02.1
8	0	2	0	1.981	1.995	+ 00.7
9	1	2	0	2.085	2.052	- 01.6
10	4	0	0	2.088	2.094	+ 00.3
11	3	1	0	2.162	1.991	- 07.9
12	2	2	0	2.268	2.208	- 02.7
13	5	0	0	2.300	2.309	+ 00.4
14	0	3	0	2.354	2.397	+ 01.8
15	0	0	1	2.372	2.372	+ 00.0
16	4	1	0	2.382	2.291	- 03.9
17	1	0	1	2.429	2.431	+ 00.1
18	1	3	0	2.472	2.438	- 01.4
19	5	1	0	2.504	2.486	- 00.7
20	0	1	1	2.543	2.529	- 00.5
21	6	0	0	2.575	2.551	- 01.0
22	2	0	1	2.602	2.589	- 00.8
23	2	3	0	2.609	2.641	+ 01.2
24	1	1	1	2.660	2.668	+ 00.3

optimal ordering of the sample points. For these examples the ordering was generated by the values of the maximum likelihood ratio confidence bounds (see below) associated with the sample points. Since this ordering is asymptotically optimal (for large λ_i's), it can be expected to produce good results for sample points well removed from the origin. The values of the confidence bounds calculated using (2.7) and the approximations obtained by the proposed algorithm are shown for these examples in Table 3.

In Table 3 the tendency of the computed values of $t(x)$ to be slightly larger than the algorithm values may be due to the fact that the ordering used to compute the former is non-optimal. Formal application of the algorithm may occasionally result in the selection of nearly equal values a_i and a_j in step (b). When this happens, improved results may be obtained by first reducing the dimension k by combining the nearly equal a_i's and then applying the algorithm. This method was used for the two cases in Table 3 marked by (†). The algorithm values for all cases where $a = (.14,.16,.70)$ or $a = (.15,.41,.44)$ do not differ substantially from the values which would be obtained by reducing to the $k = 2$ case by combining the nearly equal a_i's. Similarly, the values for the cases where $a = (.32,.33,.35)$ can be nearly reproduced by multiplying the $k = 1$ bound by $\bar{a} = .333$.

The number of sample points appearing in the ordering before a given level of $t(x)$ is reached increases rapidly as k increases.

Table 3. Several Examples Comparing the Algorithm Values With the Exact Bounds Based on the Ordering Generated by the $m\ell rb$.

$\alpha = .05$

a_1 a_2 a_3	x_1 x_2 x_3	Position in Ordering	$t(x)$	Algorithm Value $t^*(x)$	$m\ell rb$
.20,.30,.50	2, 2, 1	50	3.329	3.273	3.070
	5, 0, 2	100	4.068	3.957	3.798
	6, 6, 0	200	4.887	4.789	4.708
	9, 1, 3	400	5.980	5.892	5.795
.14,.16,.70	5, 4, 0	50	3.026	3.084†	2.218
	2,10, 0	100	3.424	3.531†	2.921
	2, 5, 1	200	4.407	4.086	3.729
.15,.41,.44	13, 0, 0	50	3.245	3.223	2.980
	5, 2, 1	100	4.028	3.860	3.699
	3, 3, 2	200	4.861	4.738	4.559
.32,.33,.35	0, 3, 2	50	3.492	3.518	3.256
	1, 5, 1	100	4.339	4.332	4.076
	1, 5, 3	200	5.198	5.234	4.993

†Reduction to k=2 case by combining a_1 and a_2.

Hence for $k > 2$ the algorithm may be applied to sample points positioned quite far along in the ordering without requiring values beyond the scope of Table 1, as is seen in the examples of Table 3.

The results detailed in Tables 2 and 3 suggest that the application of the proposed algorithm together with the combining of nearly equal a_i's when indicated will nearly always produce confidence bounds for $\theta(\lambda)$ subject to relative errors not exceeding 10 percent and often much less. Furthermore, the lower confidence bound for reliability $r(x)$ given by (1.6) will exhibit a much smaller relative error since the relative error in approximations for $t(x)$ applies only to the difference between the lower bound and one; a quantity which at worst is of the order of 1/10 in the contemplated applications.

In situations where very large sample sizes are available, it may happen that the observed numbers of failures exceed the limits of Table 1 even though the system reliability is high. For such cases the maximum likelihood ratio bound (mℓrb) may be used. This approximatic confidence bound is obtained in the usual way from the maximum likelihood ratio statistic for testing the hypothesis $H_0 : \theta(\lambda) = \theta_0$ versus all alternatives. The corresponding one-sided confidence bound may be shown (see Johns 1975) to be determined as follows: Let $\hat{\mu}$ be the positive real root less than 1/(largest a_i for which $X_i > 0$) of the equation

$$\chi^2_{1,2\alpha} = 2 \sum_{i=1}^{k} X_i \left\{ \frac{a_i\hat{\mu}}{1-a_i\hat{\mu}} + \log(1-a_i\hat{\mu}) \right\} , \tag{4.5}$$

where $\chi^2_{1,2\alpha}$ is the upper $100(2\alpha)$-th percentage point of the chi-squared distribution with one degree of freedom. Then the quantity

$$\hat{t}(X) = \sum_{i=1}^{k} a_i X_i/(1-a_i\hat{\mu}) \tag{4.6}$$

is the approximate upper $1-\alpha$ level confidence bound for $\theta(\underline{\lambda})$. As $\max(\lambda_1,\lambda_2,\ldots,\lambda_n) \to \infty$, the m$\ell$rb $\hat{t}(X)$ may be shown (see Johns 1975) to be asymptotically equivalent to

$$\tilde{t}(X) = \sum_{i=1}^{k} a_i X_i + z_\alpha \left(\sum_{i=1}^{k} a_i^2 X_i \right)^{\frac{1}{2}} , \tag{4.7}$$

where z_α is the 100α-th percentage point of the standard normal distribution.

Neither of these approximate bounds is useful for sample points near the origin in the usual orderings, but $\hat{t}$ given by (4.6) becomes sufficiently precise for application to sample points beyond the scope of Table 1. A comparison of $t(x_1,x_2)$ and the corresponding mℓrb for the last (i.e., the 100-th) points in each of the optimal orderings for the cases covered in Table 1 is given in Table 4.

Table 4. $m\ell rb/t(x_1,x_2)$ for the 100-th Point in Each Ordering

	a_1								
α	.05	.10	.15	.20	.25	.30	.35	.40	.45
.01	.62	.95	.97	.97	.98	.97	.98	.97	.99
.05	.72	.88	.96	.95	.98	.97	.99	.95	.96
.10	.89	.95	.95	.96	.97	1.00	.97	.96	.98

These results suggest that the $m\ell rb$ possesses satisfactory precision for sample points beyond those listed in Table 1 for $k = 2$ whenever $a_1 \geq .10$. The last column of Table 3 giving the $m\ell rb$ values for the examples considered illustrates the fact that for $k > 2$ the $m\ell rb$ tends to underestimate the correct value of the bound for sample points within the range that can be dealt with using Table 1 and the algorithm.

Two potential sources of error for the lower confidence bound on system reliability remain to be discussed. They are (i) the Poisson approximation to the binomial distribution of the observed failures, and (ii) the approximation for reliability given in (1.2) and reflected in the formula (1.6) for the lower bound $r(x)$. It is intuitively clear from (1.1) ff. that the "worst case" for the Poisson approximation should occur when $k = 1$; since to match a given $k = 1$ level of reliability, say $1-p$, by a $k > 1$ case, we must have $p \cong \Sigma_{i=1}^{k} p_i$, so that the p_i's must be smaller than p which tends to improve the Poisson approximation.

For the case k = 1 the familiar upper confidence bounds for a single Poisson parameter apply and the actual coverage probabilities for the proposed method (3.1) can be computed for any n and p from tables of the binomial distribution. The results of several such calculations are shown in Table 5.

Table 5. "Worst Case" Analysis (k= 1). Minimum Coverage Probabilities for r(x).

Reliability	$1-\alpha$		
$q = 1 - p$	.90	.95	.99
.95	.906	.954	.991
.90	.912	.958	.992
.80	.924	.965	.995
.70	.936	.972	.997

These values suggest that the approximations operate to make the proposed confidence bounds slightly conservative. It is interesting to observe that the minimum coverage probabilities are not drastically different from the nominal values, even for a true reliability as low as .70.

5. Acknowledgments and Remarks Concerning the Computations

The computations for Table 1 were performed on a Digital Equipment PDP-11/34 running under a UNIX operating system. The production program for these computations (506 lines) was written in the "C" language by the author. The method used for the

computation of the upper bounds was an implementation of (2.7); and as has been noted, the sample points were ordered by the two-step prospective sequential method. The tabled results were subjected to various checks to insure that the correct global maxima were found in each case.

The computations involved in obtaining the one-step look-ahead results and the tree analysis associated with the semi-Bayesian results discussed in Section 3 were done on an IBM 370/168 machine using Fortran programs developed by David Pasta. These programs compute the upper bounds by a somewhat more complicated method used in the earlier phases of the study and detailed in Johns (1975).

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The basic problem of determining objective (frequentistic) confidence bounds for the reliability of a series system based on failure data from tests of the independent components is addressed. The notion of confidence bounds based on orderings imposed on the sample space is exploited, and certain optimality considerations are incorporated. Advantage is taken of the simplifications resulting from the use of the Poisson approximation for data from highly reliable components. Tables of exact confidence bounds are produced for the case of two-component systems. These bounds are computed using sample orderings

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generated sequentially by a two-stage, prospective optimization procedure. A generalization of the Lindstrom-Madden technique is proposed for using the tables to find confidence bounds for systems consisting of more than two components with differing sample sizes.